Outline

1. Review
2. Today’s Goals
3. Cauchy-Euler Equations
4. Spring-Mass Systems with Undamped Motion
Learned how to solve nonhomogeneous linear differential equations using the method of Undetermined Coefficients.
The general solution to a linear nonhomogeneous differential equation is

\[ y_g = y_h + y_p \]

Where \( y_h \) is the solution to the corresponding homogeneous DE and \( y_p \) is any particular solution.
The Guessing Rule

For a constant-coefficient nonhomogeneous linear DE, the form of $y_p$ is a linear combination of all linearly independent functions that are generated by repeated differentiation of $g(x)$. 
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**Example:** Find the general solution to

$$y'' - y' = 4$$
The Fix to the Duplication Problem

When the natural guess for a particular solution duplicates a homogeneous solution, multiply the guess by $x^n$, where $n$ is the smallest positive integer that eliminates the duplication.
Today’s Goals

1. Learn how to solve Cauchy-Euler Equations.
2. Learn how to model spring/mass systems with undamped motion.
Goal: To solve homogeneous DEs that are not constant-coefficient.
Cauchy-Euler Equations

**Goal:** To solve homogeneous DEs that are not constant-coefficient.

**Definition**

Any linear differential equation of the form

\[ a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = g(x) \]

is a **Cauchy-Euler equation**.
The 2nd Order Case

Try to solve

$$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + \ldots cy = 0$$

by substituting $y = x^m$. 
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If $m_1$ and $m_2$ are distinct real roots to $am(m - 1) + bm + c = 0$, then the general solution to this DE is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$
For a higher order homogeneous Cauchy-Euler Equation, if \( m \) is a root of multiplicity \( k \), then

\[
x^m, \ x^m \ln(x), \ldots, x^m (\ln(x))^{k-1}
\]

are \( k \) linearly independent solutions.
For a higher order homogeneous Cauchy-Euler Equation, if $m$ is a root of multiplicity $k$, then

$$x^m, \quad x^m \ln(x), \quad \ldots, \quad x^m (\ln(x))^{k-1}$$

are $k$ linearly independent solutions.

**Example:** What is the solution to

$$x^3 y''' + xy' - y = 0$$
Conjugate Complex Roots

Given the DE

\[ ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + \ldots cy = 0 \]

If \( am(m - 1) + bm + c = 0 \) has complex conjugate roots \( \alpha + i\beta \) and \( \alpha - i\beta \), then the general solution is

\[ y_g = x^\alpha [c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x))] \]
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Example: Solve \( 25x^2 y'' + 25xy' + y = 0 \)
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A mass $m$ is attached to its free end, the amount of stretch $s$ depends on the mass.
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**Hooke’s Law:** The spring exerts a restoring force \( F \) opposite to the direction of elongation and proportional to the amount of elongation.

\[
F = ks
\]

**Note:** \( k \) is called the spring constant
Newton’s Second Law

The weight \( W = mg \) is balanced by the restoring force \( ks \) at the equilibrium position. \( mg = ks \)
Newton’s Second Law

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2. If we displace from equilibrium by distance \(x\) the restoring force becomes \(k(x + s)\).
Newton’s Second Law

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2. If we displace from equilibrium by distance \(x\) the restoring force becomes \(k(x + s)\).

Assuming free motion, **Newton’s Second Law** states

\[
m \frac{d^2x}{dt^2} = -k(s + x) + mg = -kx
\]
Question: What are the solutions to

\[ m \frac{d^2 x}{dt^2} + kx = 0? \]
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\[ m \frac{d^2x}{dt^2} + kx = 0? \]

If \( \omega^2 = \frac{k}{m} \) then the solutions are

\[ x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \]