Outline

1. Review

2. Today’s Goals

3. Spring/Mass Systems with Damped Motion
Review for Last Time

1. Learned how to solve Cauchy-Euler Equations.
2. Learned how to model spring/mass systems with undamped motion.
Cauchy-Euler Equations

Definition

Any linear differential equation of the form

\[ a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = g(x) \]

is a **Cauchy-Euler equation**.

Note: These are **not** constant coefficient.
Higher Order DEs and Repeated Roots

For a higher order homogeneous Cauchy-Euler Equation, if \( m \) is a root of multiplicity \( k \), then

\[ x^m, \ x^m \ln(x), \ ... \ , \ x^m (\ln(x))^{k-1} \]

are \( k \) linearly independent solutions.
Undamped Spring-Mass Systems

By Newton’s Second Law and Hooke’s Law, the following D.E. models an undamped mass-spring system

\[ m \frac{d^2x}{dt^2} = -kx \]

where \( k \) is the spring constant, \( m \) is the mass placed at the end of the spring and \( x(t) \) is the position of the mass at time \( t \).
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**Example:** A force of 400 newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10 m/sec. Find the equation of motion.
Today’s Goals

1. Learn how to model spring/mass systems with damped motion.
2. Learn how to model spring/mass systems with driven motion.
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\[
\frac{d^2 x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m} x = 0
\]

is now our model, where \( m \) is the mass, \( k \) is the spring constant, \( \beta \) is the damping constant and \( x(t) \) is the position of the mass at time \( t \).
Changing Variables

Let

\[ 2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega^2 = \frac{k}{m}. \]
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and the roots of the Aux. Equation become

\[ m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2} \quad \text{and} \quad m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2} \]
Case 1: Overdamped

If $\lambda^2 - \omega^2 > 0$ the system is **overdamped** since $\beta$ is large when compared to $k$. In this case the solution is

$$x = e^{-\lambda t} \left( c_1 e^{\sqrt{\lambda^2 - \omega^2} t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2} t} \right).$$
Case 2: Critically Damped

If $\lambda^2 - \omega^2 = 0$ the system is **critically damped** since a slight decrease in the damping force would result in oscillatory motion. In this case the solution is

$$x = e^{-\lambda t}(c_1 + c_2 t)$$
Case 3: Underdamped

If $\lambda^2 - \omega^2 < 0$ the system is **underdamped** since $k$ is large when compared to $\beta$. In this case the solution is

$$x = e^{-\lambda t} \left( c_1 \cos(\sqrt{\lambda^2 - \omega^2} \, t) + c_2 \sin(\sqrt{\lambda^2 - \omega^2} \, t) \right).$$
Example

A 4 foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers a damping force equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 ft/sec.
When an external force $f(t)$ acts on the mass on a spring, the equation for our model of motion becomes

$$\frac{d^2 x}{dt^2} = -\frac{\beta}{m} \frac{dx}{dt} - \frac{k}{m} x + \frac{f(t)}{m}$$
Driven Motion

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$$\frac{d^2x}{dt^2} = -\frac{\beta}{m} \frac{dx}{dt} - \frac{k}{m}x + \frac{f(t)}{m}$$

or in the language of $\lambda$ and $\omega$,

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = \frac{f(t)}{m}$$
Example

When a mass of 2 kg is attached to a spring whose constant is 32 N/m, it comes to rest at equilibrium position. Starting at $t = 0$ a force of $f(t) = 65e^{-2t}$ is applied to the system. In the absence of damping, find the equation of motion.
Example

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What is the amplitude of the oscillation after a very long time?