Math 240: Vector Calc. Review

Ryan Blair

University of Pennsylvania

Thursday March 3, 2011
1. Review
2. Today’s Goals
3. Vector-Valued Functions
4. Del, Div, Curl, Grad
Review for Last Time

1. Learned how to model spring/mass systems with damped motion.
2. Learned how to model spring/mass systems with driven motion.
Spring/Mass Example

Suppose 16 N of force stretches a spring 1 meter from equilibrium. If a mass of 4 kg is attached to the spring and subjected to a driving force given by \( f(t) = \cos(t) \), then find the equation that models the position of the system in the absence of a damping force.
Today’s Goals

1. Review vector valued functions.
2. Review del, grad, curl and div.
3. Review line integrals.
Vector-Valued Functions

Definition

Vectors whose components are functions of a parameter $t$ are called **vector-valued** functions.

\[
\mathbf{r}(t) = \langle f(t), g(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j}
\]

\[
\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{i}
\]
Vector-Valued Functions

Definition

Vectors whose components are functions of a parameter \( t \) are called **vector-valued** functions.

\[
r(t) = \langle f(t), g(t) \rangle = f(t)i + g(t)j
\]

\[
r(t) = \langle f(t), g(t), h(t) \rangle = f(t)i + g(t)j + h(t)i
\]

**Example:** \( r(t) = \langle \cos(t), \sin(t), t \rangle \)
Vector-Valued Functions

**Definition**

Vectors whose components are functions of a parameter $t$ are called **vector-valued** functions.

$$r(t) = < f(t), g(t) > = f(t)i + g(t)j$$

$$r(t) = < f(t), g(t), h(t) > = f(t)i + g(t)j + h(t)i$$

**Example:** $r(t) = < \cos(t), \sin(t), t >$

**Important:** These are the parameterized curves we will integrate along.
Derivative of a Vector-Valued Function

Definition

If \( r(t) = \langle f(t), g(t), h(t) \rangle \) where \( f, g, \) and \( h \) are differentiable, then

\[
r'(t) = \langle f'(t), g'(t), h'(t) \rangle
\]
Vector-Valued Functions

Derivative of a Vector-Valued Function

Definition

If \( r(t) = \langle f(t), g(t), h(t) \rangle \) where \( f, g, \) and \( h \) are differentiable, then

\[
r'(t) = \langle f'(t), g'(t), h'(t) \rangle
\]

Theorem

*Chain Rule* If \( r \) is a differentiable vector function and \( s = u(t) \) is a differentiable scalar function, then

\[
\frac{dr}{dt} = \frac{ds}{ds} \frac{ds}{dt} = r'(s)u'(t).
\]
Integrating Vector-Valued Functions

**Theorem**

If $f$, $g$ and $h$ are integrable and $r(t) = \langle f(t), g(t), h(t) \rangle$, then

\[
\int r(t) \, dt = \langle \int f(t) \, dt, \int g(t) \, dt, \int h(t) \, dt \rangle
\]

\[
\int_a^b r(t) \, dt = \langle \int_a^b f(t) \, dt, \int_a^b g(t) \, dt, \int_a^b h(t) \, dt \rangle
\]
The differential operator \textbf{del} is given by

\[ \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \]
The differential operator \textbf{del} is given by

\[
\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}
\]

Given a scalar function \( f(x, y, z) \) we can form the \textit{gradient of f} using \textbf{del}.

\[
\text{grad}(f) = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}
\]
Del and Grad

The differential operator **del** is given by

\[ \nabla = \nabla_i + \nabla_j + \nabla_k \]

\[ \nabla_f = \begin{aligned} \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \end{aligned} \]

Given a scalar function \( f(x, y, z) \) we can form the gradient of \( f \) using del.

\( \nabla f \) points in the direction of greatest change of \( f \).
Del and Grad

The differential operator \textbf{del} is given by

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$$

Given a scalar function $f(x, y, z)$ we can form the \textbf{gradient of $f$} using del.

$$\text{grad}(f) = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

$\nabla f$ points in the direction of greatest change of $f$.

\textbf{Example:} Guess the gradient of $f(x, y, z) = xyz$ at $(1, 1, 1)$ by interpreting the function as volume of a box.
Div and Curl

Definition

The **curl** of a vector field \( F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \) is the vector field

\[
curl(F) = \nabla \times F = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\mathbf{k}
\]
Div and Curl

Definition

The **curl** of a vector field \( F = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \) is the vector field

\[
\text{curl}(F) = \nabla \times F = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\]

Definition

The **divergence** of a vector field \( F = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \) is given by the scalar function

\[
\text{div}(F) = \nabla \cdot F = \frac{\partial P}{\partial x} \mathbf{i} + \frac{\partial Q}{\partial y} \mathbf{j} + \frac{\partial R}{\partial z} \mathbf{k}
\]
Div and Curl

Definition
The **curl** of a vector field \( F = Pi + Qj + Rk \) is the vector field

\[
curl(F) = \nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})i + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})j + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})k
\]

Definition
The **divergence** of a vector field \( F = Pi + Qj + Rk \) is given by the scalar function

\[
div(F) = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

Curl measures the tendency of a vector field to rotate. Divergence measures the tendency of a vector field to expand or contract.
Compositions

It is important to note

1. \( \text{grad}(\text{scalar function}) = \text{vector field} \)
2. \( \text{div}(\text{vector field}) = \text{scalar function} \)
3. \( \text{curl}(\text{vector field}) = \text{vector field} \)
Line Integrals in 2D

If \( G(x,y) \) is a scalar valued function and \( C \) is a smooth curve in the plane defined by the parametric equations \( x = f(t) \) and \( y = g(t) \) where \( a \leq t \leq b \) then we can define the following line integrals

1. \[ \int_C G(x, y) \, dx = \int_a^b G(f(t), g(t)) \, f'(t) \, dt \]
2. \[ \int_C G(x, y) \, dy = \int_a^b G(f(t), g(t)) \, g'(t) \, dt \]
3. \[ \int_C G(x, y) \, ds = \int_a^b G(f(t), g(t)) \sqrt{(f'(t))^2 + (g'(t))^2} \, dt \]
If $G(x,y,z)$ is a scalar valued function and $C$ is a smooth curve in 3-space defined by the parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$ where $a \leq t \leq b$ then we can define the following line integrals

1. $\int_C G(x, y, z)dx = \int_a^b G(f(t), g(t), h(t))f'(t)dt$
2. $\int_C G(x, y, z)dy = \int_a^b G(f(t), g(t), h(t))g'(t)dt$
3. $\int_C G(x, y, z)dz = \int_a^b G(f(t), g(t), h(t))h'(t)dt$
4. $\int_C G(x, y, z)ds = \int_a^b G(f(t), g(t), h(t))\sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}dt$