Math 240: Systems of Linear Differential Equations

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Outline

1. Review
2. Today's Goals
3. Linear Systems
4. Solutions to Linear Systems
Review of Last Time

Divergence Theorem

1. Outlined the proof of the divergence theorem.
2. Learned when and how to apply the divergence theorem.
Divergence Theorem

Let $D$ be a nice region in 3-space with nice boundary $S$ oriented outward. Let $F$ be a nice vector field. Then

$$\int \int_S (F \circ n) dS = \int \int \int_D \text{div}(F) dV$$

where $n$ is the unit normal vector to $S$. 

**Theorem**

Let $D$ be a nice region in 3-space with nice boundary $S$ oriented outward. Let $F$ be a nice vector field. Then
Today’s Goals

Combine linear algebra and differential equations to study systems of differential equations.

1. Define systems of differential equations
2. Develop the notion of Linear Independence.
3. Develop the notion of General Solution.
The dynamics of predictor and prey populations are modeled by the Lotka-Volterra equations

\[
\frac{dx}{dt} = x(a - by)
\]

\[
\frac{dy}{dt} = -y(c - dx)
\]
An Example of a System of D.E.s

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Where \(x(t)\) is the population of prey at time \(t\) and \(y(t)\) is the population of predators at time \(t\).
An Example of a System of D.E.s

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Where \(x(t)\) is the population of prey at time \(t\) and \(y(t)\) is the population of predators at time \(t\).

This is a **non-linear** system
Linear systems

Definition
The following is a **first order system**

\[
\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + f_1(t)
\]

\[
\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + f_2(t)
\]

\[
\vdots
\]

\[
\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + f_n(t)
\]

Where each \( x_i \) is a function of \( t \).
Every n-th order linear differential equation can be written as an $n \times n$ first order system.
Examples of First Order Systems

Every n-th order linear differential equation can be written as an $n \times n$ first order system.

**Example** Write $y'' - 3y' + 2y = 0$ as a first order system.
Solutions

Definition

Given a system $X' = AX + F$ a **solution vector** is an $n \times 1$ column matrix with differential functions as entries that satisfies the system.
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Definition

The following is an **initial value problem** for a first order system $X' = AX + F$ and $X(t_0) = X_0$
Linear Systems

Solutions

Definition

Given a system $X' = AX + F$ a solution vector is an $n \times 1$ column matrix with differential functions as entries that satisfies the system.

Definition

The following is an initial value problem for a first order system $X' = AX + F$ and $X(t_0) = X_0$

Note: As long as everything in sight is continuous on an interval $I$ containing $t_0$, then there exists a unique solution to the above IVP.
Supperposition Principle

Theorem

(Supperposition Principle) Linear combinations of solution vectors are again solution vectors.
Supperposition Principle

**Theorem**

(Supperposition Principle) Linear combinations of solution vectors are again solution vectors.

**Definition**

Solution vectors $X_1, X_2, \ldots, X_k$ are **linearly independent** if

$$c_1X_1 + c_2X_2 + \ldots + c_nX_k = 0$$

implies $c_1 = c_2 = \ldots = c_n = 0$. 
General Solutions to Homogeneous Systems

Theorem

Let $X_1, \ldots, X_n$ be a linearly independent set of solutions to a $n \times n$ first order homogeneous linear system, then the general solution is

$$X = c_1 X_1 + c_2 X_2 + \ldots + c_n X_n$$

where the $c_i$ are arbitrary constants.
General Solutions to Homogeneous Systems

Theorem

Let $X_1, ..., X_n$ be a linearly independent set of solutions to a $n \times n$ first order homogeneous linear system, then the general solution is

$$X = c_1 X_1 + c_2 X_2 + ... + c_n X_n$$

where the $c_i$ are arbitrary constants.

Note: Assuming everything in sight is differentiable, general solutions always exist.
Theorem

Let $X_p$ be a particular solution to a non-homogeneous first order linear system and $X_h$ be the general solution to the associated homogeneous equation, then the general solution is given by

$$X = X_p + X_h$$
The Wronskian

Theorem

Let $X_1, X_2, \ldots, X_n$ be $n$ solution vectors to a homogeneous system on an interval $I$. They are linearly independent if and only if their Wronskian is non-zero for every $t$ in the interval.