Outline

1. Review
2. Today’s Goals
3. Distinct Eigenvalues
4. Repeated Eigenvalues
5. Complex Eigenvalues
Review of Last Time

1. Defined systems of differential equations
2. Developed the notion of Linear Independence.
3. Developed the notion of General Solution.
Linear systems

Definition

The following is a first order system

\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n} + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n} + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn} + f_n(t)
\end{align*}
\]

Where each \( x_i \) is a function of \( t \).
The Wronskian

**Theorem**

Let $X_1, X_2, \ldots, X_n$ be $n$ solution vectors to a homogeneous system on an interval $I$. They are linearly independent if and only if their **Wronskian** is non-zero for every $t$ in the interval.
1. Be able to solve constant coefficient systems.
Given a constant coefficient, linear, homogeneous, first-order system
\[ X' = AX \]
our intuition prompts us to guess a solution vector of the form
\[ X = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = Ke^{\lambda t} \]
Guessing a Solution

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our intuition prompts us to guess a solution vector of the form

\[ X = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = Ke^{\lambda t} \]

Hence, we can find such a solution vector iff \( K \) is an eigenvector for \( A \) with eigenvalue \( \lambda \).
General Solution with Distinct Real Eigenvalues

**Theorem**

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be $n$ distinct real eigenvalues of the $n \times n$ coefficient matrix $A$ of the homogeneous system $X' = AX$, and let $K_1, K_2, \ldots, K_n$ be the corresponding eigenvectors. Then the general solution on $(-\infty, \infty)$ is

$$X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + \ldots + c_n K_n e^{\lambda_n t}$$

where the $c_i$ are arbitrary constants.
Repeated Eigenvalues

In a $n \times n$ linear system there are two possibilities for an eigenvalue $\lambda$ of multiplicity 2.

1. $\lambda$ has two linearly independent eigenvectors $K_1$ and $K_2$.
2. $\lambda$ has a single eigenvector $K$ associated to it.
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In the first case, there are linearly independent solutions $K_1e^{\lambda t}$ and $K_2e^{\lambda t}$.
Repeated Eigenvalues

In a $n \times n$ linear system there are two possibilities for an eigenvalue $\lambda$ of multiplicity 2.

1. $\lambda$ has two linearly independent eigenvectors $K_1$ and $K_2$.
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In the first case, there are linearly independent solutions $K_1 e^{\lambda t}$ and $K_2 e^{\lambda t}$.

In the second case, there are linearly independent solutions $K e^{\lambda t}$ and $[K t e^{\lambda t} + P e^{\lambda t}]$

where we find $P$ be solving $(A - \lambda I)P = K$
Real and Imaginary Parts of a Matrix

Given an $n \times m$ matrix $A$ with complex entries, 

$Re(A)$ is the real $n \times m$ matrix of the purely real entries in $A$ and 

$Im(A)$ is the real $n \times m$ matrix of purely imaginary entries of $A$. 
Theorem

Let $\lambda = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix $A$ in a homogeneous linear system $X' = A X$, and $K$ be the corresponding eigenvector. Then

$$X_1 = \left[ \text{Re}(K) \cos(\beta t) - \text{Im}(K) \sin(\beta t) \right] e^{\alpha t}$$

$$X_2 = \left[ \text{Im}(K) \cos(\beta t) + \text{Re}(K) \sin(\beta t) \right] e^{\alpha t}$$

are linearly independent solutions to $X' = A X$ on $(-\infty, \infty)$. 