1. Find the general solution to the following DE.

\[ x^2y'' - xy' + 2y = 0 \]

2. Find the general solution to the following DE.

\[ y'' - 2y' - 3y = \cos(x) \]

3. Evaluate the (unoriented) surface integral \( \int \int_S G(x, y, z) dS \) given \( G(x, y, z) = xz^3 \) and \( S \) is the portion of the cone \( z = \sqrt{x^2 + y^2} \) inside the cylinder \( x^2 + y^2 = 1 \).

4. (a) Find a number \( r \) such that the following differential equation

\[ x^2y'' + (x - 1)y' - y = 0 \]

has a power series solution of the form

\[ y(x) = x^r \cdot \left( 1 + \sum_{n=1}^{\infty} a_n x^n \right). \]

(b) For the number \( r \) you gave in (a), give a formula for the coefficients \( a_n \)'s which determines these coefficients recursively, and find \( a_1 \) and \( a_2 \).

5. Let \( B \) be the \( 3 \times 3 \) matrix

\[
B = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & -1 \\
0 & 1 & 2
\end{pmatrix}.
\]

(a) Find the characteristic polynomial \( \det(T \cdot I_3 - B) \) of \( B \).

(b) Is \( B \) diagonalizable? If so, find an invertible matrix \( C \) such that \( C^{-1} \cdot B \cdot C \) is a diagonal matrix. If not, explain why such a matrix \( C \) does not exist.

(c) (extra) Find a formula for the powers \( B^n \) of \( B \). Compute the exponential \( e^B \) of \( B \) and also the matrix-valued function \( e^{tB} \).

6. Let \( B \) be the \( 3 \times 3 \) matrix

\[
B = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & -1 \\
0 & 1 & 2
\end{pmatrix}.
\]
Find the general solution of the system of linear ordinary differential equations

\[
\frac{d}{dt} \vec{u}(t) = B \cdot \vec{u}(t), \quad \text{where} \quad \vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}
\]

7. Consider the matrix \( B \) with a parameter \( a \in \mathbb{C} \),

\[
B(a) = \begin{pmatrix} -a & a-1 \\ a & -a \end{pmatrix}.
\]

Determine all values of the parameter \( a \) such that the matrix \( B(a) \) is not diagonalizable.

8. Suppose that a function \( x(t) \) on \( \mathbb{R} \) satisfies the differential equation

\[
\frac{d^2x}{dt^2} + 6 \frac{dx}{dt} - 7 x(t) = 0.
\]

In addition we have \( x(0) = 1 \) and there exists a constant \( C > 0 \) such that

\[
|x(t)| \leq C \quad \text{for all} \ t \in \mathbb{R}.
\]

Determine this function \( x(t) \).

9. Find all solutions of the differential equation

\[
\left[ x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} - 1 \right] u(x) = x^3
\]

such that \( u(1) = 5 \) and \( \lim_{x \to 0^+} u(x) = 0 \)

10. Find a non-trivial homogeneous linear ordinary differential equation satisfied by the function \( x^2 \cdot \log x \).

11. Let \( S \) be the surface

\[
S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 4 \}
\]

oriented by the unit normal vector field

\[
\vec{N}(x, y, z) := \frac{1}{2}(x \vec{i} + y \vec{j} + z \vec{k}) \quad \text{for all} \ (x, y, z) \in S.
\]

Let \( S_r \) be the surface

\[
S_r = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 4, \ x \geq 0 \},
\]

oriented by the restriction of \( \vec{N} \) to \( S_r \).
(a) Compute the oriented surface integral
\[ \iint_S \left( \text{curl} \left( x^5 z \vec{i} + xyz \vec{j} + e^{yz} \vec{k} \right) + z \vec{k} \right) \cdot \vec{N} \, dA. \]

(This integral is also written as
\[ \iint_S \left( \text{curl} \left( x^5 z \vec{i} + xyz \vec{j} + e^{yz} \vec{k} \right) + z \vec{k} \right) \cdot \vec{N} \, dS, \]

where \( dA \) is replaced by \( dS \).)

(b) Compute the oriented surface integral
\[ \iint_{S_r} \left( \text{curl} \left( x^5 z \vec{i} + xyz \vec{j} + e^{yz} \vec{k} \right) + z \vec{k} \right) \cdot \vec{N} \, dA. \]

(c) Compute the oriented surface integral
\[ \iint_{S_r} y \vec{i} \cdot \vec{N} \, dA = \iint_{S_r} y \vec{i} \cdot dS \]

12. Show that the functions \( x, \cos x, \sin x \) are linearly independent. In other words, if \( a, b, c \) are real numbers such that
\[ ax + b \cos x + c \sin x = 0 \quad \text{for all} \ x \in \mathbb{R}, \]
then \( a = b = c = 0. \)

(Hint: Every value of \( x \) gives a linear relation between \( a, b \) and \( c \).)

13. True/False questions. Let \( A \) be a \( 4 \times 4 \) matrix with real entries such that \( \det(T \cdot I_4 - A) = (T - 1)^2 (T + 1)^2 \). For each of the following statements, determine whether it is true or false.

(a) \( A^2 \) is diagonalizable.

(b) If \( A^2 \) is diagonalizable then \( A^2 = I_4 \).

(c) \( A - I_4 \) is an invertible matrix.

(d) \( A^2 - I_4 \) is an invertible matrix.

(e) \( \dim \left( \{ \vec{x} \in \mathbb{R}^4 \mid A^2 \cdot \vec{x} = \vec{x} \} \right) = 2. \)
14. (extra) Let $S$ be the surface obtained from the curve
\[ C := \{(x, z) \in \mathbb{R}^2 \mid z = (x - 2)(1 - x), \ 1 \leq x \leq 2\} \]
on the $(x, z)$-plane by rotating $C$ about the $z$-axis. In the cylindrical coordinates $(r, \theta, z)$, points on $S$ satisfies
\[ z = (r - 2)(1 - r), \quad 1 \leq r \leq 2. \]
Let $\vec{N}$ be the continuous unit normal vector field on $S$ such that $\vec{N}(\frac{3}{2}, 0, \frac{1}{4}) = \vec{k}$. Orient the surface $S$ by $\vec{N}$. Compute the oriented surface integral
\[ \iint_S \vec{k} \cdot \vec{N} \, dA \quad (= \iint_S \vec{k} \cdot \vec{N} \, dS) \]

15. (extra) Consider the following ordinary differential equation with a parameter $a \in \mathbb{R}$,
\[ [(x^3 + ax^2 + ax + 1) \frac{d^2}{dx^2} + ax(x + 1) \frac{d}{dx} + 1] u(x) = 0. \]
For all but a finite number of real numbers $a$, the above differential equation has no irregular singularity at all points of $\mathbb{R}$, in the sense that for any $x_0 \in \mathbb{R}$ the differential equation is either ordinary at $x = x_0$ or has a regular singular point at $x = x_0$. Find the exceptional values of the parameter $a$ for which the differential equation has a irregular singular point.

16. (extra) Let $S := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 25, \ x + y + z \geq 1\}$, oriented by the unit normal vector field
\[ \vec{N}(x, y, z) = \frac{1}{5}(x\vec{i} + y\vec{j} + z\vec{k}), \quad \text{for all } (x, y, z) \in S. \]
Let $C = \partial S$ be the boundary of $S$, the circle
\[ \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 25, \ x + y + z = 1\}. \]
Let $\vec{t}$ be the unit tangent vector field on $C$ giving $C$ the orientation so that Stokes theorem holds.

(a) Determine the value $\vec{t}(-3, 4, 0)$ of the vector field $\vec{t}$ at the point $(-3, 4, 0) \in C$.
(b) Compute the oriented line integral
\[ \oint_C \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2} \, d\vec{r}, \]
where $C$ is oriented by the tangent vector field $\vec{t}$ on $C$. \\
