Outline

1. Today’s Goals
2. Distinct Eigenvalues
3. Complex Eigenvalues
4. Repeated Eigenvalues
Today’s Goals

1. Solve linear systems of differential equations with Complex Eigenvalues.
Theorem

Let $A \in M_n(\mathbb{R})$. If $A$ has $n$ linearly independent eigenvectors $v_1, v_2, ..., v_n$, with real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ (not necessarily distinct), then the general solution to $x' = Ax$ on any interval is

$$X = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + ... + c_n v_n e^{\lambda_n t}$$

Exercise: Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}$$
Theorem

Let \( \lambda = a + bi \) be a complex eigenvalue of \( A \) with eigenvectors \( v_1, \ldots, v_k \) where \( v_j = r_j + is_j \). Then the \( 2k \) real valued linearly independent solutions to \( x' = Ax \) are:

\[
e^{at}(\sin(bt)r_1 + \cos(bt)s_1), \ldots, e^{at}(\sin(bt)r_k + \cos(bt)s_k)
\]

and

\[
e^{at}(\cos(bt)r_1 - \sin(bt)s_1), \ldots, e^{at}(\cos(bt)r_k - \sin(bt)s_k)
\]
Repeated Eigenvalues

In a $n \times n$, constant-coefficient, linear system there are two possibilities for an eigenvalue $\lambda$ of multiplicity 2.

1. $\lambda$ has two linearly independent eigenvectors $K_1$ and $K_2$.
2. $\lambda$ has a single eigenvector $K$ associated to it.
Repeated Eigenvalues

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In the first case, there are linearly independent solutions $K_1 e^{\lambda t}$ and $K_2 e^{\lambda t}$. 
Repeated Eigenvalues

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1. $\lambda$ has two linearly independent eigenvectors $K_1$ and $K_2$.
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In the first case, there are linearly independent solutions $K_1 e^{\lambda t}$ and $K_2 e^{\lambda t}$.

In the second case, there are linearly independent solutions $K e^{\lambda t}$ and $[K e^{\lambda t} + P e^{\lambda t}]$

where we find $P$ be solving $(A - \lambda I)P = K$. $P$ is a generalized eigenvector.
In the second case, there are linearly independent solutions $Ke^{\lambda t}$ and $[Kte^{\lambda t} + Pe^{\lambda t}]$ where we find $P$ be solving $(A - \lambda I)P = K$.

**Exercise:** Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}$$
How Bad Can it Get?

In general, you will only be asked to solve systems $X' = AX$ if the multiplicity of the eigenvalues of $A$ is at most 1 more than the number of linearly independent eigenvectors for that value. In this case you need to find at most one vector $P$ such that $(A - \lambda I)P = K$
In general, you will only be asked to solve systems $X' = AX$ if the multiplicity of the eigenvalues of $A$ is at most 1 more than the number of linearly independent eigenvectors for that value. In this case you need to find at most one vector $P$ such that $(A - \lambda I)P = K$

**Exercise:** Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$