Outline

1. Today’s Goals

2. Solutions to homogeneous equations

3. Solutions to nonhomogeneous equations

4. Solutions to constant coefficient homogeneous equations
Today’s Goals

Understand the form of solutions to the following types of higher order, linear differential equations

1. Initial Value Problems
2. Homogeneous and Nonhomogeneous Equations.
Differential equations

Definition
A differential equation is any equation involving a function, its derivatives.

Definition
A solution to a differential equation is any function that satisfies the equation.
A Few Famous Differential Equations

1. Einstein’s field equation in general relativity
2. The Navier-Stokes equations in fluid dynamics
3. Verhulst equation - biological population growth
4. The Black-Scholes PDE - models financial markets
Higher Order Initial Value Problems

Definition

A **nth-order linear differential equation** is

\[
\text{Solve}: \quad a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)
\]
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\]

an **nth-order initial value problem** (IVP) is the above equation together with the following constraint

**Subject to**: \( y(x_0) = y_0, \quad y'(x_0) = y_1, \quad ... \), \( y^{(n-1)}(x_0) = y_{n-1} \)
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If \( g(x) = 0 \), then we say the differential equation is \textbf{homogeneous}. 
Existence and Uniqueness

Theorem

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x),$ and $g(x)$ be continuous on an interval $I$, and let $a_n(x) \neq 0$ for every $x$ in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial value problem exists on the interval and is unique.
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**Example:** Does the following IVP have a unique solution? If so, on what intervals?

$$xy''' + y'' - y' - \cos(x)y = 9$$

with $y(2) = 0$, $y'(2) = 0$ and $y''(2) = 0$
Solutions as a subspace

Theorem

(The Superposition Principle) The set of solutions to an $n$th-order homogeneous differential equation on an interval $I$ form an $n$-dimensional vector subspace of $C^n(I)$. A basis for this space is called a fundamental set.

Example: Find the fundamental set for $x'' + x = 0$ using your intuition from calculus.
Review

Definition

Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possess at least $n - 1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of the functions.
General Solutions to Nonhomogeneous Linear D.E.s

Theorem
Let $y_p$ be any particular solution of the nonhomogeneous linear $n$th-order differential equation on an interval $I$. Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions to the associated homogeneous differential equation. Then the general solution to the nonhomogeneous equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x) + y_p$$

where the $c_i$ are arbitrary constants.
A Motivating Example

Our goal is to solve **constant coefficient** linear homogeneous differential equations.
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In this case, we get \( e^{mx}(am^2 + bm + c) = 0 \). There are three possibilities for the roots of a quadratic equation.
If \( am^2 + bm + c \) has distinct roots \( m_1 \) and \( m_2 \), then the general solution to \( ay'' + by' + cy = 0 \) is

\[
y = c_1 e^{m_1 x} + c_2 e^{m_2 x}
\]
Case 2: Repeated Roots

If \( am^2 + bm + c \) has a repeated root \( m_1 \), then the general solution to 
\( ay'' + by' + cy = 0 \) is 

\[
y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}
\]
Case 3: Complex Roots

If $am^2 + bm + c$ has complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$
Auxiliary Equations

Given a linear homogeneous constant-coefficient differential equation

\[ a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... a_1 \frac{dy}{dx} + a_0 y = 0, \]

the Auxiliary Equation is

\[ a_n m^n + a_{n-1} m^{n-1} + ... a_1 m + a_0 = 0. \]
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\[ a_n m^n + a_{n-1} m^{n-1} + \ldots a_1 m + a_0 = 0. \]

The Auxiliary Equation determines the general solution.
If $m$ is a real root of the auxiliary equation of multiplicity $k$ then $e^{mx}, xe^{mx}, x^2 e^{mx}, \ldots, x^{k-1} e^{mx}$ are linearly independent solutions.
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If $(\alpha + i\beta)$ and $(\alpha + i\beta)$ are roots of the auxiliary equation of multiplicity $k$ then $e^{\alpha x} \cos(\beta x), xe^{\alpha x} \cos(\beta x), \ldots, x^{k-1} e^{\alpha x} \cos(\beta x)$ and $e^{\alpha x} \sin(\beta x), xe^{\alpha x} \sin(\beta x), \ldots, x^{k-1} e^{\alpha x} \sin(\beta x)$ are linearly independent solutions.