Math 240: Solving Constant Coefficient Linear Differential Equations

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University of Pennsylvania

Friday November 30, 2012
Outline

1. Today’s Goals

2. Solutions to constant coefficient homogeneous equations

3. Undetermined Coefficients
Today’s Goals

1. Solving arbitrary order Constant Coefficient Linear Differential Equations.
2. Use the method of undetermined coefficients to solve nonhomogeneous Constant Coefficient Linear Differential Equations.
Case 3: Complex Roots

If \( am^2 + bm + c \) has complex roots \( m_1 = \alpha + i\beta \) and \( m_2 = \alpha - i\beta \), then the general solution to \( ay'' + by' + cy = 0 \) is

\[
y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)
\]
Auxiliary Equations

Given a linear homogeneous constant-coefficient differential equation

\[ a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0, \]

the Auxiliary Equation is

\[ a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0 = 0. \]
Auxiliary Equations

Given a linear homogeneous \textbf{constant-coefficient} differential equation

\[ a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0, \]

the \textbf{Auxiliary Equation} is

\[ a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0 = 0. \]

The Auxiliary Equation determines the general solution.
If $m$ is a real root of the auxiliary equation of multiplicity $k$ then $e^{mx}, xe^{mx}, x^2 e^{mx}, \ldots, x^{k-1} e^{mx}$ are linearly independent solutions.
**General Solution from the Auxiliary Equation**

1. If $m$ is a real root of the auxiliary equation of multiplicity $k$ then $e^{mx}, xe^{mx}, x^2e^{mx}, \ldots, x^{k-1}e^{mx}$ are linearly independent solutions.

2. If $(\alpha + i\beta)$ and $(\alpha + i\beta)$ are a roots of the auxiliary equation of multiplicity $k$ then $e^{\alpha x} \cos(\beta x), xe^{\alpha x} \cos(\beta x), \ldots, x^{k-1}e^{\alpha x} \cos(\beta x)$ and $e^{\alpha x} \sin(\beta x), xe^{\alpha x} \sin(\beta x), \ldots, x^{k-1}e^{\alpha x} \sin(\beta x)$ are linearly independent solutions.
The Method of Undetermined Coefficients

Given a nonhomogeneous differential equation

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots a_1 y' + a_0 y = g(x) \]

where \( a_n, a_{n-1}, \ldots, a_0 \) are constants.
Given a nonhomogeneous differential equation

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x) \]

where \( a_n, a_{n-1}, \ldots, a_0 \) are constants.

1. Step 1: Solve the associated homogeneous equation.
The Method of Undetermined Coefficients

Given a nonhomogeneous differential equation

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x) \]

where \( a_n, a_{n-1}, \ldots, a_0 \) are constants.

1. **Step 1:** Solve the associated homogeneous equation.
2. **Step 2:** Find a particular solution by analyzing \( g(x) \) and making an educated guess.
The Method of Undetermined Coefficients

Given a nonhomogeneous differential equation

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots a_1 y' + a_0 y = g(x) \]

where \( a_n, a_{n-1}, \ldots, a_0 \) are constants.

1. Step 1: Solve the associated homogeneous equation.
2. Step 2: Find a particular solution by analyzing \( g(x) \) and making an educated guess.
3. Step 3: Add the homogeneous solution and the particular solution together to get the general solution.
Guessing Particular Solutions

\[ g(x) \]

*constant*  

Guess
Undetermined Coefficients

Guessing Particular Solutions

\[ g(x) \quad \text{Guess} \]

constant \[ A \]
Undetermined Coefficients

Guessing Particular Solutions

g(x)  
constant
3x^2 − 2  

<table>
<thead>
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Guessing Particular Solutions

\[ g(x) \]

**constant**

\[ 3x^2 - 2 \]

**Guess**

\[ A \]

\[ Ax^2 + Bx + C \]
Guessing Particular Solutions

\[ g(x) \]

- \( \text{constant} \)
- \( 3x^2 - 2 \)
- \( \text{Polynomial of degree } n \)

**Guess**

- \( A \)
- \( Ax^2 + Bx + C \)
**Guessing Particular Solutions**

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Guessing Particular Solutions

$g(x)$

*constant*

$3x^2 - 2$

*Polynomial of degree $n$*

$\cos(4x)$

**Guess**

$A$

$Ax^2 + Bx + C$

$A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0$
Guessing Particular Solutions

- **g(x)**
  - constant
  - $3x^2 - 2$
  - Polynomial of degree $n$
  - $\cos(4x)$

- **Guess**
  - $A$
  - $Ax^2 + Bx + C$
  - $A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0$
  - $A \cos(4x) + B \sin(4x)$
Guessing Particular Solutions

\[ g(x) \]
\[ constant \]
\[ 3x^2 - 2 \]
\[ Polynomial \ of \ degree \ n \]
\[ \cos(4x) \]
\[ A \cos(nx) + B \sin(nx) \]

**Guess**

- \( A \)
- \( A x^2 + B x + C \)
- \( A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0 \)
- \( A \cos(4x) + B \sin(4x) \)
Guessing Particular Solutions

\( g(x) \)

- constant
- \( 3x^2 - 2 \)
- Polynomial of degree \( n \)
- \( \cos(4x) \)
- \( \cos(nx) + B\sin(nx) \)

**Guess**

- \( A \)
- \( Ax^2 + Bx + C \)
- \( A_nx^n + A_{n-1}x^{n-1} + \ldots + A_0 \)
- \( A\cos(4x) + B\sin(4x) \)
- \( A\cos(nx) + B\sin(nx) \)
Guessing Particular Solutions

\[
g(x) \quad \text{Guess}
\]
\[
\begin{align*}
\text{constant} & : A \\
3x^2 - 2 & : Ax^2 + Bx + C \\
\text{Polynomial of degree } n & : A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0 \\
\cos(4x) & : A \cos(4x) + B \sin(4x) \\
A \cos(nx) + B \sin(nx) & : A \cos(nx) + B \sin(nx) \\
e^{4x} & : \\
\end{align*}
\]
### Undetermined Coefficients

#### Guessing Particular Solutions

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Guessing Particular Solutions

\[ g(x) \]

\[ constant \]
\[ 3x^2 - 2 \]

\[ Polynomial of degree n \]
\[ A_{n-1}x^{n-1} + \ldots + A_0 \]

\[ A\cos(4x) + B\sin(4x) \]

\[ A\cos(nx) + B\sin(nx) \]

\[ Ae^{4x} \]

\[ x^2 e^{5x} \]

\[ Ae^{2x}\sin(4x) + Be^{2x}\cos(4x) \]

\[ 3x\sin(5x) \]

\[ (Ax + B)\sin(5x) + (Cx + D)\cos(5x) \]

\[ xe^{2x}\cos(3x) \]

\[ (Ax + B)e^{2x}\sin(3x) + (Cx + D)e^{2x}\cos(3x) \]
The form of $y_p$ is a linear combination of all linearly independent functions that are generated by repeated differentiation of $g(x)$. 
A Problem

Solve $y'' - 5y' + 4y = 8e^x$ using undetermined coefficients.
The solution

When the natural guess for a particular solution duplicates a homogeneous solution, multiply the guess by $x^n$, where $n$ is the smallest positive integer that eliminates the duplication.