Math 240: Eigenvalues and Linear Transformations of $\mathbb{R}^2$

Ryan Blair

University of Pennsylvania

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Outline

1. Invertible Linear Transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$

2. Eigenvalue and Eigenvector
Today’s Goals

1. Know how to decompose linear transformations of $\mathbb{R}^2$ into stretches, reflections and shears.
2. Know how to calculate eigenvalues and eigenvectors.
How to build any invertible linear transformation

Theorem

Any linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ with invertible matrix is obtained by composing reflections, stretches and shears.
Row Operations as Linear Transformations for 2x2 Matrices

Reflections

\[ P_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

Stretching

\[ M_1(k) = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2(k) = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \]

Shearing

\[ A_{21}(k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad A_{12}(k) = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \]
Definition

Let $\lambda$ be a scalar, $x$ be a $n \times 1$ column vector and $A$ be a $n \times n$ matrix. Any nontrivial vector that solves $Ax = \lambda x$ is called an eigenvector. If $Ax = \lambda x$ has a non-trivial solution, $\lambda$ is an eigenvalue.
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Key idea: Eigenvectors are vectors sent to scalar copies of themselves under the linear map corresponding to $A$. 
How to find Eigenvalues

To find eigenvalues we want to solve $Ax = \lambda x$ for $\lambda$.

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$Ax - \lambda x = 0$

$(A - \lambda I_n)x = 0$
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Hence, to find the eigenvalues, we solve the polynomial equation $det(A - \lambda I_n) = 0$ called the characteristic equation.
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$det(A - \lambda I_n) = 0$ called the characteristic equation.

For each eigenvalue $\lambda$, solve the linear system $(A - \lambda I_n)x = 0$ to find the eigenvectors.