Outline

1. Today’s Goals
2. Diagonalizability Theorems
3. Linear Systems
4. Solutions to Linear Systems
5. Distinct Eigenvalues
Today’s Goals

Combine linear algebra and differential equations to study systems of differential equations.

1. Define systems of differential equations
2. Develop the notion of Linear Independence.
3. Develop the notion of General Solution.
Theorem

A $n \times n$ matrix is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

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If an $n \times n$ matrix has $n$ distinct eigenvalues, then it is diagonalizable.
Diagonalizability Theorems

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Note: Not all diagonalizable matrices have $n$ distinct eigenvalues.
Definition

A system of differential equations is of the form

\[ x'(t) = A(t)x(t) + b(t) \]

Where \( A(t) \) is an \( n \times n \) matrix of functions, both \( x(t) \) and \( b(t) \) are \( n \times 1 \) matrices of functions and \( x'(t) \) is the \( n \times 1 \) matrix of derivatives of entries in \( x(t) \).

A solution

Example:

\[ x_1'(t) = \cos(t)x_1(t) - \sin(t)x_2(t) \]
\[ x_2'(t) = \sin(t)x_1(t) + \cos(t)x_2(t) \]
Solutions

Definition

Given a system \( x'(t) = A(t)x(t) + b(t) \) a **solution vector** is a \( n \times 1 \) column matrix with differentialable functions as entries that satisfies the system.
Solutions

Definition

Given a system $x'(t) = A(t)x(t) + b(t)$ a solution vector is an $n \times 1$ column matrix with differentialable functions as entries that satisfies the system.

Definition

The following is an initial value problem for a first order system $x'(t) = A(t)x(t) + b(t)$ and $x(t_0) = x_0$.
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The following is an initial value problem for a first order system \( x'(t) = A(t)x(t) + b(t) \) and \( x(t_0) = x_0 \)

Note: As long as everything in sight is continuous on an interval \( I \) containing \( t_0 \), then there exists a unique solution to the above IVP.
Solutions to Linear Systems

Let $V_n(t)$ be the set of $n \times 1$ matrices with entries consisting of functions. $V_n(t)$ is a vector space under the natural operations of vector addition and scalar multiplication.

**Theorem**

*Solutions to homogeneous systems of differential equations of the form*

$$x'(t) = A(t)x(t)$$

*form a vector subspace of $V_n(t)$.*

**Definition**

Given an $n \times n$ homogeneous solution $x'(t) = A(t)x(t)$, a fundamental solution is a set of $n$ linearly independent solutions.
The Wronskian

**Theorem**

Let $X_1, X_2, \ldots, X_n$ be $n$ solution vectors to a homogeneous system on an interval $I$. They are linearly independent if and only if their **Wronskian** is non-zero for every $t$ in the interval.
Guessing a Solution

Given a constant coefficient, linear, homogeneous, first-order system

\[ x' = Ax \]

our intuition prompts us to guess a solution vector of the form

\[ x = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = K e^{\lambda t} \]
Given a constant coefficient, linear, homogeneous, first-order system

\[ x' = Ax \]

our intuition prompts us to guess a solution vector of the form

\[
    \begin{pmatrix}
        k_1 \\
        k_2 \\
        \vdots \\
        k_n
    \end{pmatrix}
\]

\[ e^{\lambda t} = Ke^{\lambda t} \]

Hence, we can find such a solution vector iff \( K \) is an eigenvector for \( A \) with eigenvalue \( \lambda \).
Theorem

Let $\lambda_1$, $\lambda_2$, ..., $\lambda_n$ be $n$ distinct real eigenvalues of the $n \times n$ coefficient matrix $A$ of the homogeneous system $X' = AX$, and let $K_1$, $K_2$, ..., $K_n$ be the corresponding eigenvectors. Then the general solution on $(-\infty, \infty)$ is

$$X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + ... + c_n K_n e^{\lambda_n t}$$

where the $c_i$ are arbitrary constants.
Distinct Eigenvalues

General Solution with Distinct Real Eigenvalues

**Theorem**

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be $n$ distinct real eigenvalues of the $n \times n$ coefficient matrix $A$ of the homogeneous system $X' = AX$, and let $K_1, K_2, \ldots, K_n$ be the corresponding eigenvectors. Then the general solution on $(-\infty, \infty)$ is

$$X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + \ldots + c_n K_n e^{\lambda_n t}$$

where the $c_i$ are arbitrary constants.

**Exercise:** Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$$