Math 240: Spring-Mass Systems

Ryan Blair

University of Pennsylvania

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Outline

1. Today’s Goals
2. Review
3. Spring-Mass Systems with Undamped Motion
4. Spring/Mass Systems with Damped Motion
Today’s Goals

1. Learn how to solve spring/mass systems.
The Method of Undetermined Coefficients

To solve a nonhomogeneous constant coefficient linear differential equation
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1. Step 1: Solve the associated homogeneous equation.
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2. Step 2: Find a particular solution by making a guess based on $g(x)$.
The Method of Undetermined Coefficients

To solve a nonhomogeneous constant coefficient linear differential equation

1. Step 1: Solve the associated homogeneous equation.
2. Step 2: Find a particular solution by making a guess based on $g(x)$.
3. Step 3: Add the homogeneous solution and the particular solution together to get the general solution.
Spring-Mass Systems with Undamped Motion

A flexible spring of length $l_0$ is suspended vertically from a rigid support.
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A mass $m$ is attached to its free end, the amount of stretch $L_0$ depends on the mass.
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**Hooke’s Law:** The spring exerts a restoring force $F_s$ opposite to the direction of elongation and proportional to the amount of elongation.

$$F_s = -kL_0$$
Newton’s Second Law

The force due to gravity ($F_g = mg$) is balanced by the restoring force $-kL_0$ at the equilibrium position.

$mg = kL_0$
Newton’s Second Law

1. The force due to gravity \( F_g = mg \) is balanced by the restoring force \(-kL_0\) at the equilibrium position.
\[ mg = kL_0 \]

2. If we displace from equilibrium by distance \( y \) the restoring force becomes \( k(y + L_0) \).
Newton’s Second Law

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Assuming free motion, **Newton’s Second Law** states

\[
m\frac{d^2y}{dt^2} = -k(L_0 + y) + mg = -ky
\]
Solutions to Undamped Spring Equation

**Question:** What are the solutions to

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If \( \omega_0^2 = \frac{k}{m} \) then the solutions are

\[ y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t). \]

Example: A force of 400 newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10 m/sec. Find the equation of motion.
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\[ m \frac{d^2 y}{dt^2} + c \frac{dy}{dt} + ky = 0 \]

is now our model, where \( m \) is the mass, \( k \) is the positive spring constant, \( c \) is the positive damping constant and \( y(t) \) is the position of the mass at time \( t \).
Changing Variables

Let

\[ 2\lambda = \frac{c}{m} \quad \text{and} \quad \omega_0^2 = \frac{k}{m}. \]
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and the roots of the Aux. Equation become

\[ m_1 = -\lambda + \sqrt{\lambda^2 - \omega_0^2} \quad \text{and} \quad m_2 = -\lambda - \sqrt{\lambda^2 - \omega_0^2} \]
Case 1: Overdamped

If $\lambda^2 - \omega_0^2 > 0$ the system is **overdamped** since $c$ is large when compared to $k$. In this case the solution is

$$y = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2 - \omega_0^2} t} + c_2 e^{-\sqrt{\lambda^2 - \omega_0^2} t}).$$
Case 2: Critically Damped

If $\lambda^2 - \omega_0^2 = 0$ the system is critically damped since a slight decrease in the damping force would result in oscillatory motion. In this case the solution is

$$y = e^{-\lambda t}(c_1 + c_2 t)$$
Case 3: Underdamped

If \( \lambda^2 - \omega_0^2 < 0 \) the system is underdamped since \( k \) is large when compared to \( c \). In this case the solution is

\[
y = e^{-\lambda t}(c_1 \cos(\sqrt{\omega_0^2 - \lambda^2} t) + c_2 \sin(\sqrt{\omega_0^2 - \lambda^2} t))
\]
Example

A 4 meter spring measures 8 meters long after a force of 16 newtons acts to it. A mass of 8 kilograms is attached to the spring. The medium through which the mass moves offers a damping force equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 meters/sec.