Math 240: Surface Integrals

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Outline

1. Today’s Goals
2. Review
3. Surface Integrals
Today’s Goals

1. Review of surface area for a parameterized surfaces.
2. Be able to evaluate Surface Integrals.
Surface Area of a Parameterized Surface

Given a parametrization $\mathbf{X} : D \rightarrow \mathbb{R}^3$ such that $X(s, t) = (x(s, t), y(s, t), z(s, t))$.

The surface area of $S = \mathbf{X}(D)$ is equal to

$$\int \int_D \| T_s \times T_t \| \, dsdt$$
**Scalar Surface Integrals**

**Definition**

Let \( \mathbf{X} : D \rightarrow \mathbb{R}^3 \) be a smooth parametrized surface, where \( D \) is a region in \( \mathbb{R}^2 \). Let \( f : \mathbf{X}(D) \rightarrow \mathbb{R} \) be a continuous function. Then the **scalar surface integral** of \( f \) along \( \mathbf{X} \) is

\[
\int \int_{\mathbf{X}} f \, dS = \int \int_{D} f(\mathbf{X}(s, t)) \parallel T_s \times T_t \parallel \, dsdt
\]

**Example** Find \( \int \int_{S} x^2 + y^2 \, dS \), where \( S \) is the outward oriented lateral surface of the cylinder of radius \( a \) and height \( h \) whose axis is the \( z \)-axis.
Vector Surface Integrals

Definition

Let $\mathbf{X} : D \rightarrow \mathbb{R}^3$ be a smooth parametrized surface, where $D$ is a region in $\mathbb{R}^2$. Let $\mathbf{F}(x, y, z)$ be a continuous function. Then the vector surface integral (or Flux) of $\mathbf{F}$ along $\mathbf{X}$ is

$$\int \int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \int \int_{D} \mathbf{F}(\mathbf{X}(s, t)) \cdot \mathbf{N}(s, t) dsdt$$

where $\mathbf{N}(s, t) = T_s \times T_t$.

Example

Find the flux of $\mathbf{F} = xi + yj + zk$ across the surface $S$ consisting of the triangular region of the plane $2x - 2y + z = 2$ that is cut out by the coordinate planes. Use an upward-pointing normal to orient $S$. 
Orientation

Definition
An **orientable** surface has two sides that can be painted red and blue resp.

Definition
If a parameterized surface $S$ is orientable, then an **orientation** is a choice of one of two normal vectors.

$$N_1 = T_s \times T_t$$

or

$$N_2 = T_t \times T_s = -N_1$$