Math 240: Div, Curl and Line Integrals

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Friday January 13, 2012
Outline

1. Review
2. Today’s Goals
3. Del, Div, Curl, Grad
Review for Last Time

1. Reviewed the definition of vector valued functions and their derivatives.
2. Reviewed the definition of and the calculation of partial derivatives.
Partial Derivative Example

Find \( \frac{\partial w}{\partial x} \) if \( w = y^{\ln(x)} \cos(xz) \).
Today’s Goals

1. Define and calculate \( \nabla, \nabla \cdot, \nabla \times, \text{ and div} \).
2. Review line integrals.
A 3-dimensional vector field is a map from $\mathbb{R}^3$ to $\mathbb{R}^3$ denoted by

$$F(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$$

where $f(x, y, z)$, $g(x, y, z)$ and $h(x, y, z)$ are scalar valued functions.
Definition

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Similarly, a 2-dimensional vector field is of the form

$$F(x, y) = \langle f(x, y), g(x, y) \rangle.$$
Del and Grad

**Motivation:** We want an alternative notion of derivative of a function from $\mathbb{R}^2$ or $\mathbb{R}^3$ into $\mathbb{R}$.
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The differential operator \textbf{del} is given by

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$$
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Given a scalar function $f(x, y, z)$ we can form the **gradient of $f$** using del.

$$\text{grad}(f) = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$
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$\nabla f$ points in the direction of greatest change of $f$. 

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Example: Guess the gradient of $f(x, y, z) = xyz$ at $(1, 1, 1)$ by interpreting the function as volume of a box.
Motivation: Given a vector field we want to make quantitative the notion of expansion and contraction.
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**Definition**

The **divergence** of a vector field \( F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \) is given by the scalar function

\[
div(F) = \nabla \cdot F = \frac{\partial P}{\partial x}\mathbf{i} + \frac{\partial Q}{\partial y}\mathbf{j} + \frac{\partial R}{\partial z}\mathbf{k}
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Divergence measures the tendency of a vector field to expand or contract.
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Definition

The **curl** of a vector field \( F = Pi + Qj + Rk \) is the vector field

\[
\text{curl}(F) = \nabla \times F = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)i + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)j + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)k
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Definition

The curl of a vector field $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the vector field

$$\text{curl}(F) = \nabla \times F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

The magnitude of Curl is the intensity of the rotation about a point. The direction of Curl is the axis of maximal rotation about a point.
Question: What is curl in dimension 2?
**Curl in Dimension 2**

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**Theorem (Green’s Theorem)**

Suppose $C$ is a piecewise smooth simple closed curve bounding a region $R$. If $P$, $Q$, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous on $R$, then

$$\int_C P\,dx + Q\,dy = \int \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\,dA,$$

where $C$ is oriented counterclockwise.
Domain and Range

It is important to note

1. \( \text{grad}(\text{scalar function}) = \text{vector field} \)
2. \( \text{div}(\text{vector field}) = \text{scalar function} \)
3. \( \text{curl}(\text{vector field}) = \text{vector field} \)
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**Homework:** If \( F \) is a 3-dimensional vector field show

\[
\text{div}(\text{curl}(F)) = 0
\]