Math 240: Double Integrals in Polar Coordinates and Green’s Theorem

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Outline

1. Today’s Goals
2. Review Problem
3. Green’s Theorem
Today’s Goals

1. Review the calculation of double integrals in polar coordinates.
2. Review Green’s Theorem.
Example: For the region $R$ bounded by $y = x$, $x + y = 4$ and $x = 0$ evaluate

$$\int \int_R x + 1 dA$$
Evaluation of Double Integrals in Polar Coordinates

Theorem

Let \( f \) be continuous on a region \( R \).

If \( R \) is Type PI, then

\[
\int \int_R f(r, \theta)\,dA = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} f(r, \theta)\,r\,dr\,d\theta
\]

If \( R \) is Type PII, then

\[
\int \int_R f(r, \theta)\,dA = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r, \theta)\,r\,d\theta\,dr
\]

If \( R \) is Type PIII, then

\[
\int \int_R f(r, \theta)\,dA = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r, \theta)\,r\,d\theta\,dr
\]
Review Problem

Change of Coordinates

If a region in the plane can be describe in polar coordinates as

\[ 0 \leq g_1(\theta) \leq r \leq g_2(\theta), \quad \alpha \leq \theta \leq \beta \]

then we have the following conversion formula

\[
\int \int_R f(x, y) dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(rcos(\theta), rsin(\theta)) r dr d\theta
\]
Change of Coordinates

If a region in the plane can be describe in polar coordinates as

$$0 \leq g_1(\theta) \leq r \leq g_2(\theta), \quad \alpha \leq \theta \leq \beta$$

then we have the following conversion formula

$$\int \int_R f(x, y) \, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(rcos(\theta), rsin(\theta)) \, r \, dr \, d\theta$$

Example Evaluate

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \sqrt{x^2 + y^2} \, dy \, dx$$
Green’s Theorem

**Theorem (Green’s Theorem)**

Suppose $C$ is a piecewise smooth simple closed curve bounding a region $R$. If $P$, $Q$, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous on $R$, then

$$\oint_C P \, dx + Q \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA,$$

where $C$ is oriented counterclockwise.