Outline

1. Properties of Determinants
2. Matrix Inverse
3. Properties of Inverses
4. Solving a Linear System Using Inverses
Using Elementary Row Operations to Find the Determinant

Suppose $B$ is obtained from $A$ by:

1. multiplying a row by a non-zero scalar $c$, then $\det(A) = \frac{1}{c} \det(B)$.
2. switching rows, then $\det(A) = -\det(B)$.
3. adding a multiple of one row to another row, then $\det(A) = \det(B)$. 
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**Example:** Find the determinant of the following matrix.

$$
\begin{pmatrix}
0 & 2 & 0 & -3 \\
3 & 0 & 2 & 5 \\
-2 & 4 & 0 & 6 \\
0 & 1 & 1 & 1
\end{pmatrix}
$$
Properties of Determinants

Theorem

If elementary row or column operations lead to one of the following conditions, then the determinant is zero.

1. *an entire row (or column) consists of zeros.*
2. *one row (or column) is a multiple of another row (or column).*
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Let $A$ and $B$ be $n \times n$ matrices and $c$ be a scalar.

1. $\det(AB) = \det(A)\det(B)$
2. $\det(cA) = c^n\det(A)$
3. $\det(A^T) = \det(A)$
Today’s Goals

1. Be able to find the inverse of a matrix or show it has no inverse.
2. Know the properties of inverses.
3. Be able to solve systems of linear equations using matrices.
Definition

An $n \times n$ matrix $A$ is **invertible** if there exists an $n \times n$ matrix $B$ such that

$$AB = BA = I_n.$$ 

In this case, $B$ is the **inverse** of $A$. 

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1. NOT every matrix is invertible.
2. A matrix that is not invertible is called **singular**.
3. If $A$ is invertible, its inverse is denoted $A^{-1}$.
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**Example:** check the the following matrices are inverses of each other.

$$\begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} -7 & 2 \\ 4 & -1 \end{pmatrix}$$
A 2 × 2 Matrix Inverse Formula

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2 × 2 matrix and $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
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A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

Exercise: Prove the above statement
How to find the inverse of an arbitrary $n \times n$ matrix $A$.

1. Form the augmented $n \times 2n$ matrix $[A|I_n]$.
2. Find the reduced row echelon form of $[A|I_n]$.
3. If $\text{rank}(A) < n$ then $A$ is not invertible.
4. If $\text{rank}(A) = n$, then the RREF form of the augmented matrix is $[I_n|A^{-1}]$. 
Inverses of Arbitrary $n \times n$ Matrices

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Find the inverse of

$$
\begin{pmatrix}
-1 & 3 & 0 \\
1 & -2 & 1 \\
0 & 1 & 2
\end{pmatrix}
$$
Properties of Inverses

1. \((A^{-1})^{-1} = A\)
2. \((cA)^{-1} = \frac{1}{c} A^{-1}\)
3. \((AB)^{-1} = B^{-1} A^{-1}\)
4. \((A^T)^{-1} = (A^{-1})^T\)
5. \(det(A^{-1}) = \frac{1}{det(A)}\)
6. \(A\) is invertible if and only if \(det(A) \neq 0\)
Let $A$ be invertible and $Ax = B$ be a linear system, then the solution to the linear system is given by

$$x = A^{-1}B$$
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**Example:** Solve the following linear system using inverses.

$$x + z = -4$$

$$x + y + z = 0$$

$$5x - y = 6$$