Outline

1. Today's Goals
2. Solving D.E.s Using Auxiliary Equations
Today's Goals

1. Use auxiliary equations to solve constant coefficient linear homogeneous D.E.s
If $y_1, y_2, \ldots, y_n$ are linearly independent solutions to a homogeneous $n$-th order linear D.E., then $c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is the general solution.
1. If $y_1, y_2, ..., y_n$ are linearly independent solutions to a homogeneous $n$-th order linear D.E., then $c_1y_1 + c_2y_2 + ... + c_ny_n$ is the general solution.

2. If $y_h$ is the homogeneous solution and $y_p$ is the particular solution to a non-homogeneous linear D.E., then $y_h + y_p$ is the general solution.
A Motivating Example

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What if we guess \( y = e^{mx} \) as a solution to \( ay'' + by' + cy = 0 \)?
A Motivating Example

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What if we guess $y = e^{mx}$ as a solution to $y'' + y' - 6y = 0$?

What if we guess $y = e^{mx}$ as a solution to $ay'' + by' + cy = 0$?

In this case, we get $e^{mx}(am^2 + bm + c) = 0$. There are three possibilities for the roots of a quadratic equation.
Case 1: Distinct Roots

If \( am^2 + bm + c \) has distinct roots \( m_1 \) and \( m_2 \), then the general solution to \( ay'' + by' + cy = 0 \) is

\[
y = c_1 e^{m_1 x} + c_2 e^{m_2 x}
\]
Case 2: Repeated Roots

If $am^2 + bm + c$ has a repeated root $m_1$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1x} + c_2 xe^{m_1x}.$$
Magic!

\[ e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots \]
$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots$

$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \ldots$
\[ e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots \]

\[ = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \ldots \]

\[ = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots) \]
Solving D.E.s Using Auxiliary Equations

Magic!

\[ e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots \]

\[ = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \ldots \]

\[ = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots) \]

\[ = \cos(\theta) + isin(\theta) \]
Case 3: Complex Roots

If \( am^2 + bm + c \) has complex roots \( m_1 = \alpha + i\beta \) and \( m_2 = \alpha - i\beta \), then the general solution to \( ay'' + by' + cy = 0 \) is

\[
y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)
\]
Auxiliary Equations

Given a linear homogeneous constant-coefficient differential equation

\[ a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1 \frac{dy}{dx} + a_0 y = 0, \]

the **Auxiliary Equation** is

\[ a_n m^n + a_{n-1} m^{n-1} + \ldots a_1 m + a_0 = 0. \]
Given a linear homogeneous constant-coefficient differential equation

\[ a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = 0, \]

the Auxiliary Equation is

\[ a_n m^n + a_{n-1} m^{n-1} + ... + a_1 m + a_0 = 0. \]

The Auxiliary Equation determines the general solution.
General Solution from the Auxiliary Equation

If $m$ is a real root of the auxiliary equation of multiplicity $k$ then $e^{mx}, xe^{mx}, x^2 e^{mx}, \ldots, x^{k-1} e^{mx}$ are linearly independent solutions.
General Solution from the Auxiliary Equation

1. If $m$ is a real root of the auxiliary equation of multiplicity $k$ then $e^{mx}, xe^{mx}, x^2 e^{mx}, \ldots, x^{k-1} e^{mx}$ are linearly independent solutions.

2. If $(\alpha + i\beta)$ and $(\alpha + i\beta)$ are roots of the auxiliary equation of multiplicity $k$ then $e^{\alpha x} \cos(\beta x), xe^{\alpha x} \cos(\beta x), \ldots, x^{k-1} e^{\alpha x} \cos(\beta x)$ and $e^{\alpha x} \sin(\beta x), xe^{\alpha x} \sin(\beta x), \ldots, x^{k-1} e^{\alpha x} \sin(\beta x)$ are linearly independent solutions.