Math 240: Undetermined Coefficients

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Outline

1. Today’s Goals
2. Review
3. Undetermined Coefficients
Use the method of undetermined coefficients to solve the nonhomogeneous differential equations.
Auxiliary Equations

Given a linear homogeneous constant-coefficient differential equation

\[ a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1 \frac{dy}{dx} + a_0 y = 0, \]

the Auxiliary Equation is

\[ a_n m^n + a_{n-1} m^{n-1} + \ldots a_1 m + a_0 = 0. \]
Auxiliary Equations

Given a linear homogeneous constant-coefficient differential equation

\[ a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1 \frac{dy}{dx} + a_0 y = 0, \]

the **Auxiliary Equation** is

\[ a_n m^n + a_{n-1} m^{n-1} + \ldots a_1 m + a_0 = 0. \]

The **Auxiliary Equation** determines the general solution.
If $m$ is a real root of the auxiliary equation of multiplicity $k$ then $e^{mx}, xe^{mx}, x^2 e^{mx}, \ldots , x^{k-1} e^{mx}$ are linearly independent solutions.
General Solution from the Auxiliary Equation

1. If \( m \) is a real root of the auxiliary equation of multiplicity \( k \) then
\[ e^{mx}, xe^{mx}, x^2 e^{mx}, \ldots, x^{k-1} e^{mx} \]
are linearly independent solutions.

2. If \((\alpha + i\beta)\) and \((\alpha + i\beta)\) are a roots of the auxiliary equation of multiplicity \( k \) then
\[ e^{\alpha x} \cos(\beta x), xe^{\alpha x} \cos(\beta x), \ldots, x^{k-1} e^{\alpha x} \cos(\beta x) \]
and
\[ e^{\alpha x} \sin(\beta x), xe^{\alpha x} \sin(\beta x), \ldots, x^{k-1} e^{\alpha x} \sin(\beta x) \]
are linearly independent solutions.
Given a nonhomogeneous differential equation

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x) \]

where \( a_n, a_{n-1}, \ldots, a_0 \) are constants.
The Method of Undetermined Coefficients

Given a nonhomogeneous differential equation

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1. Step 1: Solve the associated homogeneous equation.
The Method of Undetermined Coefficients

Given a nonhomogeneous differential equation

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where \( a_n, a_{n-1}, \ldots, a_0 \) are constants.

1. Step 1: Solve the associated homogeneous equation.
2. Step 2: Find a particular solution by analyzing \( g(x) \) and making an educated guess.
The Method of Undetermined Coefficients

Given a nonhomogeneous differential equation

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x) \]

where \( a_n, a_{n-1}, \ldots, a_0 \) are constants.

1. **Step 1:** Solve the associated homogeneous equation.
2. **Step 2:** Find a particular solution by analyzing \( g(x) \) and making an educated guess.
3. **Step 3:** Add the homogeneous solution and the particular solution together to get the general solution.
Guessing Particular Solutions

\[ g(x) \]

**Guess**

*constant*
Guessing Particular Solutions

g(x)  
\text{constant}  

\text{Guess}  
A
Guessing Particular Solutions

\[ g(x) \]

*constant\]

\[ 3x^2 - 2 \]

**Guess**

\[ A \]
Guessing Particular Solutions

\[ g(x) \]

\[ constant \]

\[ 3x^2 - 2 \]

**Guess**

\[ A \]

\[ Ax^2 + Bx + C \]
Guessing Particular Solutions

\[ g(x) \]

\begin{align*}
\text{constant} & : A \\
3x^2 - 2 & : Ax^2 + Bx + C \\
\text{Polynomial of degree } n & \\
\end{align*}
Guessing Particular Solutions

\[ g(x) \quad \text{Guess} \]

- constant: \( A \)
- \( 3x^2 - 2 \): \( Ax^2 + Bx + C \)
- \( \text{Polynomial of degree } n \): \( A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0 \)
Guessing Particular Solutions

g(x)
constant
$3x^2 - 2$
Polynomial of degree $n$
$\cos(4x)$

Guess
$A$
$Ax^2 + Bx + C$
$A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0$
Guessing Particular Solutions

\( g(x) \)  \hspace{2cm} \text{Guess}

\begin{align*}
\text{constant} & \quad A \\
3x^2 - 2 & \quad Ax^2 + Bx + C \\
\text{Polynomial of degree } n & \quad A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0 \\
\cos(4x) & \quad A \cos(4x) + B \sin(4x)
\end{align*}
Guessing Particular Solutions

\[ g(x) \]
\[ constant \]
\[ 3x^2 - 2 \]
\[ Polynomial \ of \ degree \ n \]
\[ \cos(4x) \]
\[ A \cos(nx) + B \sin(nx) \]

**Guess**

\[ A \]
\[ A x^2 + B x + C \]
\[ A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0 \]
\[ A \cos(4x) + B \sin(4x) \]
### Guessing Particular Solutions

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Guessing Particular Solutions

\( g(x) \)

\begin{align*}
\text{constant} & \quad A \\
3x^2 - 2 & \quad Ax^2 + Bx + C \\
\text{Polynomial of degree } n & \quad A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0 \\
\cos(4x) & \quad A \cos(4x) + B \sin(4x) \\
A \cos(nx) + B \sin(nx) & \quad A \cos(nx) + B \sin(nx) \\
e^{4x} & \quad A e^{4x} \\
x^2 e^{5x} & \quad (A x^2 + B x + C) e^{5x} \\
e^{2x} \cos(4x) & \quad A e^{2x} \sin(4x) + B e^{2x} \cos(4x) \\
3x \sin(5x) & \quad (A x + B) \sin(5x) + (C x + D) \cos(5x) \\
x e^{2x} \cos(3x) & \quad (A x + B) e^{2x} \sin(3x) + (C x + D) e^{2x} \cos(3x)
\end{align*}
The form of $y_p$ is a linear combination of all linearly independent functions that are generated by repeated differentiation of $g(x)$. 
A Problem

Solve \( y'' - 5y' + 4y = 8e^x \) using undetermined coefficients.
The solution

When the natural guess for a particular solution duplicates a homogeneous solution, multiply the guess by $x^n$, where $n$ is the smallest positive integer that eliminates the duplication.