Outline

1. Notes on Eigenvalues
2. Diagonalizability
Today’s Goals

1. Be able to diagonalize matrices.
2. Be able to use diagonalization to compute high powers of matrices.
Important Examples

1. A matrix may have no eigenvalues (We don’t count non-real eigenvalues)
2. A matrix may have multiple eigenvectors for a single eigenvalue.
3. A $n \times n$ matrix may not have $n$ linearly independent eigenvectors.
Definition

An $n \times n$ matrix $A$ is **diagonalizable** if there exists an $n \times n$ invertible matrix $P$ and an $n \times n$ diagonal matrix $D$ such that $P^{-1}AP = D$.

When $A$ is diagonalizable, the columns of $P$ are the eigenvectors of $A$ and the diagonal entries of $D$ are the corresponding eigenvalues.
Diagonalizability

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**Example:** Find an invertible matrix $P$ and a diagonal matrix $D$ so that $P^{-1}AP = D$.

\[
A = \begin{pmatrix}
1 & 0 & 1 \\
0 & -1 & 3 \\
0 & 0 & 2
\end{pmatrix}
\]
Diagonalizability Theorems

Theorem

A $n \times n$ matrix is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

Theorem

If an $n \times n$ matrix has $n$ distinct eigenvalues, then it is diagonalizable.
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If an \( n \times n \) matrix has \( n \) distinct eigenvalues, then it is diagonalizable.

Note: Not all diagonalizable matrices have \( n \) distinct eigenvalues.
If a matrix is diagonalizable, there is a very fast way to compute its powers.
Using Diagonalization to Find Powers

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If $A$ is diagonalizable, then

$$A^n = (PDP^{-1})^n = PD^nP^{-1}.$$
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Example: Given

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

compute $A^8$. 