Let \( F = (e^z - 1, 2y\cos z, -y^2\sin z + 3 + xe^z) \) be a vector field in \( \mathbb{R}^3 \). Compute the line integral \( \int F \cdot dr \) along the curve \( r(t) = (\cos t, \sin t, t) \) from \( t = 0 \) to \( t = \frac{\pi}{2} \).

Solution: Since this integral looks like it will be an absolute nightmare, our first intuition is to check if the vector field is conservative (i.e. equal to \( \nabla f \) for some function \( f \)). If it were, then:

\[
(e^z - 1, 2y\cos z, -y^2\sin z + 3 + xe^z) = F = \nabla f = (f_x, f_y, f_z)
\]

In particular, setting equal the first coordinates and integrating to find \( f \):

\[
f = \int f_x \, dx = \int (e^z - 1) \, dx = xe^z - x + \alpha(y, z)
\]

Here \( \alpha(y, z) \) represents all of the terms in our function \( f \) that don’t have any \( x \)'s in them. They would all be wiped out when we take the partial derivative with respect to \( x \). Likewise for the other coordinates:

\[
f = \int f_y \, dy = \int 2y\cos z \, dy = y^2\cos z + \beta(x, z)
\]

\[
f = \int f_z \, dz = \int (-y^2\sin z + 3 + xe^z) \, dz = y^2\cos z + 3z + xe^z + \gamma(x, y)
\]

Where again the functions \( \beta \) and \( \gamma \) are terms that do not depends on \( y \) and \( z \), respectively. We see that if we let \( f = xe^z + y^2\cos z - x + 3z \) it will satisfy all three of the above partial derivatives. Our line integral starts at \( p = (1, 0, 0) \) and ends at \( p = (0, 1, \frac{\pi}{2}) \). Thus \( F = \nabla f \) and our line integral can be computed by just taking the difference of \( f(p) \) and \( f(q) \):

\[
\int F \cdot dr = \int \nabla f \cdot dr = f(x, y, z) \mid_{(1,0,0)}^{(0,1,\frac{\pi}{2})} = \frac{3\pi}{2}
\]