Asymptotically Truthful Equilibrium Selection in Large Congestion Games

Ryan Rogers - Penn
Aaron Roth - Penn

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Related Work

• **Large Games**
  - Roberts and Postlewaite 1976
  - Bodoh-Creed 2013
  - Azevedo and Budish 2011

• **Incorporating a Mediator**
  - Monderer and Tennenholtz 2003, 2009
  - Ashlagi et al 2009

• **Work most closely related to ours**
  - Kearns et al 2014.
Routing Game

\[ \zeta_e(y) \]
Routing Game

- A game $\mathcal{G}$ is defined by
  - A set of $n$ players
  - A set of types $\mathcal{U} \mapsto$ source destination pair $s_i \in \mathcal{U}$.
  - A set of actions $A \mapsto$ routes for each source destination pair.
  - A cost function $c : \mathcal{U} \times A^n \to \mathbb{R}$

$$c(s_i, a) = \sum_{e \in a_i} \ell_e(y_e(a))$$
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- Players may not know each other’s type.
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  - Types may be sensitive information
- **Main Goal**: Have players play a pure strategy Nash equilibrium of the complete information game in settings of partial information.
Mediator for a Routing Game
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• A mediator is an algorithm $M : (\mathcal{U} \cup \bot)^n \rightarrow (\mathcal{A} \cup \bot)^n$. 
Weak Mediator

- Mediator cannot force people to use it
- Players need not follow its suggested action
- Players may lie to the mechanism if they choose to use it.
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Augmented Game

• Define the augmented game $G_M$ (Kearns et al 2014):
  • Action Space:
    
    \[ A' = \{(s, f) : s \in \mathcal{U} \cup \bot, f : (A \cup \bot) \rightarrow A\} \]

    \[ g_i = (s_i, f_i) \in A' \]

  • Costs for \( g' = ((s'_i, f_i))_{i=1}^n \):
    
    \[ c_M(s_i, g') = \mathbb{E}_{a \sim M(s')} [c(s_i, f(a))] \]
Good Behavior

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  - Follow the suggested action of $M \Rightarrow f_i = \text{identity map.}$
(Kearns et al 2014) Let $M : D^n \rightarrow O^n$. Then $M$ satisfies $\epsilon$-joint differential privacy if for every $s \in D^n$, for every $i \in [n]$, $s'_i \in D$ and for every $B \subseteq O^{n-1}$

$$\mathbb{P}[M(s)_{-i} \in B] \leq e^\epsilon \mathbb{P}[M(s'_i, s_{-i})_{-i} \in B]$$
If a mechanism $M : \mathbb{U} \rightarrow \mathbb{O}$ is $(\epsilon, \delta)$-differentially private and consider any function $\phi : \mathbb{U} \times \mathbb{O} \rightarrow \mathbb{A}$. Define $M' : \mathbb{U} \rightarrow \mathbb{A}$ to be $M'(s)_i = \phi(s_i, M(s))$. Then $M'$ is $(\epsilon, \delta)$-jointly differentially private.
• If a mechanism $M : \mathcal{U}^n \rightarrow O$ is $(\epsilon, \delta)$-differentially private and consider any function $\phi : \mathcal{U} \times O \rightarrow A^n$. Define $M' : \mathcal{U}^n \rightarrow A^n$ to be

$$M'(s)_i = \phi(s_i, M(s)).$$

Then $M'$ is $(\epsilon, \delta)$-joint differentially private.
Motivating Theorem

• Let $G$ be any game with costs in $[0, m]$, and let $M$ be a mediator such that

  • It is $\epsilon$-joint differentially private
  • For any set of reported types $s$, it outputs an $\eta$-approximate pure strategy Nash Equilibrium.

  Then good behavior $g^*$ is an $\eta'$-approximate ex-post equilibrium for the incomplete information game $G_M$ where $\eta' = 2m\epsilon + \eta$. 
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**Main Theorem**

- There exists such a mechanism from the motivating theorem for large congestion games.
- Further, we show that good behavior $g^*$ is an $\eta'$-approximate ex-post equilibrium for the incomplete information game $G_M$ where

$$\eta' = \tilde{O} \left( \left( \frac{m^5}{n} \right)^{1/4} \right) \rightarrow 0 \text{ as } n \rightarrow \infty$$
Large Games
Large Games

• We assume that each player cannot significantly change the cost of another player by changing her route.

\[ |\ell_e(y_e) - \ell_e(y_e + 1)| \leq \frac{1}{n} \quad \text{for } y_e \in [n] \text{ and } e \in E. \]

• The costs then satisfy for \( j \neq i \) and \( a'_j \neq a'_j \in A \)

\[ |c(s_i, (a_j, a_{-j})) - c(s_i, (a'_j, a_{-j}))| \leq \frac{m}{n}. \]
How to Construct Such a Mechanism?
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- Simulate Best Response Dynamics
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- Compute Best Responses privately
How to Construct Such a Mechanism?

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- Limit the number of times a single player can change routes.
Best Responses

- In congestion games, allowing each player to best respond given the other players routes will converge to an approximate Nash Equilibrium.
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• There can be no more than $T = \frac{mn}{\alpha}$ best responses.

• We need to only maintain a count of the number of people on every edge to compute $\alpha$-Best Responses for each player
Binary Mechanism

- Chan et al 2011 and Dwork et al 2010 give a way to obtain an online count of a sensitivity 1 stream \( \omega \in \{0, 1\}^T \) such that the output \( \hat{y}^t \) for any \( t = 1, 2, \ldots, T \) is
  - \( \epsilon \) differentially private
  - Has high accuracy to the exact count \( y^t \) for every \( t = 1, \ldots, T \)

\[
|\hat{y}^t - y^t| \leq \tilde{O}\left(\frac{1}{\epsilon}\right)
\]
Generalized Binary Mechanism

• Each of the $m$ streams are $k$-sensitive, so we get...

• $\epsilon$-differentially private counters

• With high probability $|\hat{y}_t - y_t| \leq \tilde{O}(km\epsilon) \quad \forall e \in E, t = 1, \ldots, T$
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The Gap

- After a player $i$ has made an $\alpha$-private best response, how many times must other players move before $i$ can move again? We will call this the gap $\gamma$. 

- Due to the largeness condition, each time a player does not move, her cost can increase by at most $mn$ and can only move once her cost has increased by $\alpha \gamma = \tilde{\Omega}(\alpha n^2 m)$. 

- All the players can only make $T = \tilde{O}(mn^2 \alpha)$ (with high probability). 

- A player only changes routes $k$ times $k = \tilde{O}(m^2 \alpha^2)$. 

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• A player only changes routes $k$ times

$$k = \mathcal{O} \left( \frac{m^2}{\alpha^2} \right)$$
Equilibrium Analysis of our Algorithm

• With high probability, after $T = \tilde{O}\left(\frac{mn}{\alpha}\right)$ moves by all players, no player will be able to improve her private cost by more than $\alpha$. If we set

$$\alpha = \tilde{\Theta}\left(\left(\frac{m^4}{n\epsilon}\right)^{1/3}\right)$$

then we know no player will be able to improve her actual cost by more than

$$\eta \leq \alpha + \text{Error from BM} = \tilde{O}\left(\left(\frac{m^4}{n\epsilon}\right)^{1/3}\right)$$
Equilibrium Analysis of our Algorithm

- Is it Joint Differentially Private?
Equilibrium Analysis of our Algorithm

- Is it Joint Differentially Private?
- Recall our motivating theorem that says *good* behavior is an $\eta'$-approximate ex-post equilibrium for $G_M$ and we can set $\epsilon$ (which is a parameter we control) to satisfy the following

$$
\eta' = \tilde{O}\left(\left(\frac{m^5}{n}\right)^{1/4}\right) \rightarrow 0 \ \text{as} \ n \rightarrow \infty
$$
Open Questions

- Can Nash Equilibria of the complete information game be implemented as exact ex-post or Bayes Nash Equilibria of the incomplete information game?
- Does there exist a jointly differentially private algorithm for computing approximate Nash Equilibria for general large games?