1) Suppose $S$ is the portion of the plane $x + 2y + 2z = 6$ that lies inside the first octant given by the mapping:

$$\Phi(x, y) = \left( x, y, \frac{6 - x - 2y}{2} \right)$$

and

$$\omega = xdy \wedge dz + ydz \wedge dx + 3zdx \wedge dy$$

is a 2-form over $\mathbb{R}^3$. Compute the integral of $\omega$ over $S$.

Note that

$$\int_S \omega = \int_D 3z \frac{\partial(x, y)}{\partial(x, y)} + x \frac{\partial(y, z)}{\partial(x, y)} + y \frac{\partial(z, x)}{\partial(x, y)} dxdy$$

Calculating the Jacobians, we get:

$$\begin{vmatrix} \frac{\partial(x, y)}{\partial(s, t)} & \frac{\partial(y, z)}{\partial(s, t)} & \frac{\partial(z, x)}{\partial(s, t)} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \end{vmatrix}$$

Substituting:

$$\int_S \omega = \int_D 3\left(\frac{6 - x - 2y}{2}\right) + \frac{1}{2}x + y dxdy$$

$$= \int_0^3 \int_0^{6-2y} 3\left(\frac{6 - x - 2y}{2}\right) + \frac{1}{2}x + y dxdy$$

$$= \int_0^3 \int_0^{6-2y} 9 - x - 2y + y dxdy$$

$$= \int_0^3 36 - 18y + 2y^2 dy$$

$$= \frac{45}{2}$$

2) Let $S$ be the graph of $z = x^2 + y^2 + 7$ over the disk $x^2 + y^2 \leq 9$. Let $\omega = x^2(dx \wedge dy)$. Calculate the integral of $\omega$ over $S$.

Parametrize the surface $S$ using the map $\Phi(s, t) = (s, t, s^2 + t^2 + 7)$. Then $\frac{\partial(x, y)}{\partial(s, t)} = 1$ and we get:

$$\int_S \omega = \int_D (s^2 + t^2 + 7)^2 \frac{\partial(x, y)}{\partial(s, t)} dsdtdt$$

$$= \int_D (s^2 + t^2 + 7)^2 dsdtdt$$
It will be easier to integrate over $D$ in polar coordinates, so let $s = r \cos(\theta)$ and $t = r \sin(\theta)$. The Jacobian resulting from changing to polar coordinates is $r$, so:

$$
\int_D (s^2 + t^2 + 7)^2 dsdt = \int_0^{2\pi} \int_0^3 r(r^2 + 7)^2 drd\theta
$$

$$
= \int_0^{2\pi} \frac{1251}{2} d\theta
$$

$$
= \frac{1251\pi}{2}
$$

3) Let

$$
\omega = zdy \wedge dz + (y + 1)dz \wedge dx - dx \wedge dy
$$

and let

$$
T(u, v) = (u^2 + v, u - v, uv)
$$

be a mapping from $\mathbb{R}^2 \to \mathbb{R}^3$. Find the pullback $T^*\omega$

Note that:

$$
dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv = (2u)du + 1dv
$$

$$
dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv = 1du - 1dv
$$

$$
dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = vdu + udv
$$

So:

$$
T^*\omega = uv((du - dv) \wedge (vdu + udv)) + (u - v + 1)((vdu + udv) \wedge (2udu + 1dv))
$$

$$
- (2u(du \wedge dv) \wedge (du - dv))
$$

$$
= u^2v(du \wedge dv) - uv^2(dv \wedge du) + v(u - v + 1)(du \wedge dv)
$$

$$
+ 2u^2(u - v + 1)(dv \wedge du) + 2u(du \wedge dv) - 1(dv \wedge du)
$$

$$
= \left((-2u^3 + 3u^2v - 2u^2 + uv^2 + uv + 2u - v^2 + v + 1)(du \wedge dv)\right)
$$

4) Let

$$
\omega = 2xydx + (x^2 - z^2)dy - 2yzdz
$$

(a) Check that $\omega$ is a closed form
(b) Determine whether there exists a smooth function $f(x, y, z)$ over $\mathbb{R}^3$ such that $df = \omega$. Find all $f$ satisfying this condition.

(c) Find the line integral of $\omega$ along the parametrized curve $\gamma(t) = (1-t, \sin(\pi t), t^2+2)$ for $0 \leq t \leq 1$.

(a) To check that $\omega$ is closed, we need to show that $d(\omega) = 0$. So:

$$d(\omega) = d(2xydx + (x^2 - z^2)dy - yzdz)$$
$$= (2ydx + 2xdy) \wedge dx + (2xdx - 2zdz) \wedge dy - (2zdy + 2ydz) \wedge dz$$
$$= 2x(dy \wedge dx) + 2x(dx \wedge dy) - 2z(dz \wedge dy) - 2z(dy \wedge dz)$$
$$= 0$$

(b) We need:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$
$$= \omega$$

This implies that:

$$\frac{\partial f}{\partial x} = 2xy$$
$$\frac{\partial f}{\partial y} = x^2 - z^2$$
$$\frac{\partial f}{\partial z} = -2yz$$

An $f$ that satisfies such conditions is $f = x^2y - yz^2 + c$, where $c$ is some constant.

(c) Using Stokes’s theorem, which states that:

$$\int_{\partial \gamma} \phi = \int_{\gamma} d(\phi)$$

Noting that the boundary of a line is described by its endpoints

$$\int_{\gamma} d(\phi) = \phi(<0, 0, 3>) - \phi(<1, 0, 2>)$$
$$= 0$$
5) Let
\[ \omega = (12x^2y^3 + 2y)(dx \wedge dy) \]
Check that \( \omega \) is closed and find a smooth \( \phi \) s.t. \( d\omega = \phi \).

\[
d(\omega) = d(12x^2y^3 + 2y) \wedge dx \wedge dy \\
= ((24xy^3)dx + (36x^2y^2 + 2)dy) \wedge dx \wedge dy \\
= 0 + 0 \\
= 0
\]

To find a \( \phi \) such that \( d(\phi) = \omega \), let \( \phi = A dx + B dy \). Then:
\[
d(\phi) = \left( \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right) \wedge (dx) + \left( \frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right) \wedge dy \\
= \left( \frac{\partial A}{\partial y} \right) (dy \wedge dx) + \left( \frac{\partial B}{\partial x} \right) (dx \wedge dy)
\]
Thus, we can set \( A = 0 \) and find a \( B \) such that \( \frac{\partial B}{\partial x} = 12x^2y^3 + 2y \). If we simply integrate \( 12x^2y^3 + 2y \) w.r.t. \( x \), we get:
\[ B = 4x^3y^3 + 2xy \]
This implies that a \( \phi \) such that \( d(\phi) = \omega \) is:
\[ \phi = (4x^3y^3 + 2xy)dy \]

6) Let
\[ \omega = (3y - 2yz)(dx \wedge dy) + y^2(dz \wedge dx) + 2z(dy \wedge dz) \]
Check that \( \omega \) is closed and find a smooth \( \phi \) s.t. \( d\omega = \phi \).

To check that \( \omega \) is closed:
\[
d\omega = (3dy - 2ydz) \wedge (dx \wedge dy) + 2ydy \wedge dz \wedge dx + 2dz \wedge dy \wedge dz \\
= -2y(dz \wedge dx \wedge dy) + 2y(dy \wedge dz \wedge dz) \\
= 0
\]
Now, let \( \phi = Pdx + Qdy + Rdz \). Then:

\[
d\phi = \left( \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \wedge dx \\
+ \left( \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy + \frac{\partial Q}{\partial z} dz \right) \wedge dy \\
+ \left( \frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \frac{\partial R}{\partial z} dz \right) \wedge dz
\]

\[
= \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) (dx \wedge dy) + \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) (dx \wedge dz) + \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) (dy \wedge dz)
\]

Thus, a general form of the solution will satisfy the following relationships:

\[
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 - 2yz \\
\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = y^2 \\
\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 2z
\]

To simplify, force \( R = 0 \). Then:

\[
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 - 2yz \\
\frac{\partial P}{\partial z} = y^2 \\
\frac{\partial Q}{\partial z} = -2z
\]

Let \( P = zy^2 \) and \( Q = 3yx - z^2 \). Then:

\[
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 - 2yz \\
\frac{\partial P}{\partial z} = y^2 \\
\frac{\partial Q}{\partial z} = -2z
\]

as we wanted. Thus, one such \( \phi \) s.t. \( d\phi = \omega \) is:

\[
\phi = zy^2 dx + (3yx - z^2) dy
\]