1. True or false: if ω is a k-form and k is odd, then $\omega \wedge \omega = 0$. What if k is even and $k \geq 2$?

   I claim that this statement is true. Assume that $\omega = \sum_I a_I(x) dx_I$, where $I = I_1, ..., I_n$ and each I_u represents a specific $i_{u_1}, ..., i_{u_k}$ Then, we have

$$\begin{aligned}
\omega \wedge \omega &= \sum_{c=1}^{n} \sum_{d=1}^{n} a_{I_c}(x) a_{I_d}(x) dx_{I_c} \wedge dx_{I_d} \\
&= \sum_{c=d} a_{I_c}(x) a_{I_d}(x) dx_{I_c} \wedge dx_{I_d} + \sum_{c \neq d} a_{I_c}(x) a_{I_d}(x) dx_{I_c} \wedge dx_{I_d} \\
&= \sum_{c \neq d} a_{I_c}(x) a_{I_d}(x) dx_{I_c} \wedge dx_{I_d} \\
&= \sum_{c<d} a_{I_c}(x) a_{I_d}(x) dx_{I_c} \wedge dx_{I_d} + \sum_{c>d} a_{I_c}(x) a_{I_d}(x) dx_{I_c} \wedge dx_{I_d} \\
&= \sum_{c<d} a_{I_c}(x) a_{I_d}(x) dx_{I_c} \wedge dx_{I_d} + a_{I_c}(x) a_{I_d}(x) dx_{I_d} \wedge dx_{I_c} \\
&= \sum_{c<d} a_{I_c}(x) a_{I_d}(x) (dx_{I_c} \wedge dx_{I_d} + dx_{I_d} \wedge dx_{I_c})
\end{aligned}$$

   Claim: $dx_{I_c} \wedge dx_{I_d} + dx_{I_d} \wedge dx_{I_c} = 0$.

   Proof of this claim: Let $I_c = i_1, ..., i_k$ and let $I_d = j_1, ..., j_k$, where k is odd.

   Then $dx_{I_c} \wedge dx_{I_d} = dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k} \wedge dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_k}$

   $= (-1)^{k^2} dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_k} \wedge dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k} = -dx_{I_d} \wedge dx_{I_c}$.

   Therefore $\omega \wedge \omega = 0$ when k is odd.

   The statement is false if k is even and $k \geq 2$. Counterexample: let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$.

   Then

$$\begin{aligned}
\omega \wedge \omega &= (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \\
&= dx_1 \wedge dx_2 \wedge dx_1 \wedge dx_2 + dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \wedge dx_3 \wedge dx_4 \\
&= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4
\end{aligned}$$

   And $\omega \wedge \omega \neq 0$.

2. a) Check that for any 2-simplex $\sigma = [p_0, p_1, p_2]$, we have $\partial(\partial\sigma) = 0$.

   By definition, $\partial\sigma = [p_1, p_2] - [p_0, p_2] + [p_0, p_1]$.

   Therefore $\partial(\partial\sigma) = [p_2] - [p_1] - ([p_2] - [p_0]) + [p_1] - [p_0] = [p_2] - [p_2] + [p_1] - [p_1] + [p_0] - [p_0] = 0$.

   b) Show that for any k -chain Γ, $\partial(\partial\Gamma) = 0$.

   First I will show that this is true for any k-simplex σ.

   If $\sigma = [p_0, p_1, ..., p_k]$, then by definition

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i \sigma_{-i}$$

Where $\sigma_{-i} = [p_0, p_1, ..., p_{i-1}, p_{i+1}, p_{i+2}, ..., p_k]$

Then taking the boundary of this $k-1$ chain, where we let $\{0, 1, 2, ..., i-1, i+1, i+2, ..., k\} = \{i_0, i_1, i_2, ..., i_i, ..., i_{k-1},\}$ and $\sigma_{-i,-j}$ represents first removing i then j,

$$\partial(\partial\sigma) = \sum_{i=0}^{k}\sum_{j=0}^{k-1}(-1)^i(-1)^j[p_{i_0}, p_{i_1}, ..., p_{i_{j-1}}, \hat{p_{i_j}}, p_{i_{j+1}}, ..., p_{i_{k-1}}]$$

If $j > i$, $i_j = j+1$, and if $j < i$, $i_j = j$. Thus

$$\begin{aligned}
&= \sum_{i=0}^{k-1}\sum_{j=i+1}^{k}(-1)^i(-1)^{j-1}[p_0, p_1, ..., p_{i-1}, \hat{p_i}, p_{i+1}, ..., p_{j-1}, \hat{p_j}, p_{j+1}, ..., p_k] \\
&+ \sum_{i=1}^{k}\sum_{j=0}^{i-1}(-1)^i(-1)^j[p_0, p_1, ..., p_{i-1}, \hat{p_i}, p_{i+1}, ..., p_{j-1}, \hat{p_j}, p_{j+1}, ..., p_k] \\
&= \sum_{i<j}(-1)^i(-1)^{j-1}\sigma_{i,j} + \sum_{i>j}(-1)^i(-1)^j\sigma_{i,j} = 0
\end{aligned}$$

Because in the sum, for any a,b such that $0 \leq a < b \leq k$, since we can either remove p_a or p_b first, we are adding

$$\begin{aligned}
&(-1)^a(-1)^{b-1}[p_0, p_1, ..., p_{a-1}, \hat{p_a}, p_{a+1}, ..., p_{b-1}, \hat{p_b}, p_{b+1}, ..., p_k] \\
&+(-1)^a(-1)^b[p_0, p_1, ..., p_{a-1}, \hat{p_a}, p_{a+1}, ..., p_{b-1}, \hat{p_b}, p_{b+1}, ..., p_k] = 0.
\end{aligned}$$

Therefore, $\partial(\partial)\sigma = 0$ for any k-simplex σ. Therefore, for any k - chain $\Gamma = \sigma_1 + \sigma_2 + \cdots + \sigma_n$,

$$\begin{aligned}
\partial(\partial\Gamma) &= \partial(\partial(\sigma_1 + \cdots + \sigma_n)) \\
&= \partial(\partial\sigma_1 + \partial\sigma_2 + \cdots + \partial\sigma_n) \\
&= \partial(\partial\sigma_1) + \partial(\partial\sigma_2) + \cdots + \partial(\partial\sigma_n) = 0
\end{aligned}$$

3. Rudin Chapter 10, # 17,18,24,25,26

17. Put $J^2 = \tau_1 + \tau_2$, where $\tau_1 = [0, e_1, e_1 + e_2]$, $\tau_2 = -[0, e_2, e_2 + e_1]$. Explain why it is reasonable to call J^2 the positively oriented unit square in R^2. Show that ∂J^2 is the sum of 4 oriented affine 1-simplexes. Find these. What is $\partial(\tau_1 - \tau_2)$?

If we draw a diagonal line from (0,0) to (1,1), τ_1 is the bottom half of the unit square and τ_2 is the top half of the unit square, by defintion. We want to find the orientation of these simplices. First we note that $\tau_2 = -[0, e_2, e_2 + e_1] = [0, e_2 + e_1, e_2]$

We can write $\tau_1(u) = 0 + A_1u$, $\tau_2(u) = A_2u$ as in equation (78) on page 266, where $u \in \mathbb{R}^2$. Then

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and $\det A_1 = 1$ and $\det A_2 = 1$. By the definition on page 267 of Rudin, τ_1 and τ_2 are positively oriented. Thus J^2 is positively oriented.

$$\begin{aligned}\partial J^2 &= \partial(\tau_1+\tau_2) = \partial(\tau_1)+\partial(\tau_2)\\ &= [e_1, e_1+e_2] - [0, e_1+e_2] + [0, e_1] + [e_2+e_1, e_2] - [0, e_2] + [0, e_2+e_1]\\ &= [e_1, e_1+e_2] + [0, e_1] + [e_1+e_2, e_2] + [e_2, 0]\end{aligned}$$

Thus $\partial(J^2)$ is the sum of four oriented affine 1-simplexes.

$$\partial(\tau_1-\tau_2) = \partial[0, e_1, e_1+e_2] + \partial[0, e_2, e_1+e_2]$$
$$= [e_1, e_1+e_2] - [0, e_1+e_2] + [0, e_1] + [e_2, e_1+e_2] - [0, e_1+e_2] + [0, e_2]$$
$$= [e_1, e_1+e_2] + [0, e_1] + [e_2, e_1+e_2] + [0, e_2] - 2[0, e_1+e_2]$$

18. Consider the oriented affine 3-simplex

$$\sigma_1 = [0, e_1, e_1+e_2, e_1+e_2+e_3]$$

in $\mathbb{R}^3$. Show that σ_1 (regarded as a linear transformation) has determinant 1. Thus σ_1 is positively oriented.

Since the first element (p_0) of σ_1 is the zero vector, the transformation A associated with σ_1 must satisfy for any $u \in Q^3$ (the standard 3 simplex) $\sigma_1(u) = Au$. Thus

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

A is an upper triangular matrix, so multiplying the elements on the diagonal we get that $\det A = 1$. Thus σ_1 is positively oriented.

24. Let $\omega = \sum a_i(x)dx_i$ be a 1-form of class C'' in a convex open set $E \subset \mathbb{R}^n$. Assume $d\omega = 0$ and prove that ω is exact in E by completing the following outline:

Fix $p \in E$. Define

$$f(x) = \int_{[p,x]}$$

For affine-oriented 2-simplexes $[p, x, y]$ in E, since p, x, y are in E and E is convex, the line segments connecting any two of the three points are also contained entirely within E. Therefore if we let $\sigma = [p, x, y]$, the boundary of σ is given by $\partial\sigma = [x, y] - [p, y] + [p, x]$. Since $d\omega = 0$, by Stokes' theorem

$$0 = \int_\sigma d\omega = \int_{\partial\sigma} \omega = \int_{[x,y]} \omega - \int_{[p,y]} \omega + \int_{[p,x]} \omega$$

Therefore

$$\int_{[p,y]} \omega - \int_{[p,x]} \omega = \int_{[x,y]} \omega.$$

Note that the left hand side of this equation is equal to $f(y) - f(x)$, and $[x, y]$ is the line segment from x to y. We can parametrize this line segment by $\gamma(t) = x + (y - x)t$. Since $x \in \mathbb{R}^n$, we can rewrite $\gamma(t) = (x_1 + (y_1 - x_1)t, x_2 + (y_2 - x_2)t, ..., (x_n + (y_n - x_n)t))$.

Then taking the line integral as usual, we get

$$\begin{aligned} f(y) - f(x) &= \int_{[x,y]} \omega \\ &= \sum_{i=1}^{n} \int_0^1 a_i(x + (y-x)t)\gamma_i'(t)dt \\ &= \sum_{i=1}^{n} \int_0^1 a_i(x + (y-x)t)(y_i - x_i)dt \\ &= \sum_{i=1}^{n} (y_i - x_i) \int_0^1 a_i((1-t)x + ty)dt \end{aligned}$$

for $x, y \in E$. Therefore for $x \in E$,

$$\begin{aligned} (D_i f)(x) &= \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{h \int_0^1 a_i((1-t)x + t(x + he_i))dt}{h} \\ &= \lim_{h \to 0} \int_0^1 a_i((1-t)x + t(x + he_i)) \\ &= \int_0^1 a_i(x)dt = a_i(x) \end{aligned}$$

Hence ω is exact.

25. Assume that ω is a 1-form in an open set $E \subset \mathbb{R}^n$ such that

$$\int_\gamma \omega = 0$$

for every closed curve γ in E, of class C'. Prove that ω is exact in E.

Let $\omega = \sum a_i(x)dx_i$. Want to show that there exists $f(x) such that df = \omega$.

Fix $p \in E$. E is open, so $\exists \delta > 0$ such that $N_\delta(p) \subset E$. Then for any $x, y \in N_\delta(p)$, let $f(x) = \int_{[p,x]} \omega$. Let γ be the 1-chain $\gamma = [x, y] - [p, y] + [p, x]$. So *gamma* is a closed curve, and

$$0 = \int_\gamma \omega = \int_{[x,y]-[p,y]+[p,x]} \omega = \int_{[x,y]} \omega - \int_{[p,y]} \omega + \int_{[p,x]} \omega$$

Therefore we have

$$f(y) - f(x) = \int_{[x,y]} \omega$$

when $x, y \in N_\delta(p)$. By the exact same argument as in 24 (we have the same parametrization of $[x, y]$), $(D_i f)(x) = a_i(x)$ and ω is exact.

26. Assume ω is a 1-form in $\mathbb{R}^3 - \{0\}$, of class C' and $d\omega = 0$. Prove that ω is exact in $\mathbb{R}^3 - \{0\}$.

Let $E = \mathbb{R}^3 - \{0\}$ for simplicity - note that E is open. We know that every closed continuously differentiable curve in E is the boundary of a 2-surface in E. Let γ be a closed continuously differentiable curve in E; it is the boundary of a 2-surface S in E. That is, let S be the surface such that $\partial S = \gamma$. Then by Stokes' theorem,

$$0 = \int_S 0 = \int_S d\omega = \int_{\partial S} \omega$$

So

$$\int_{\partial S} \omega = \int_\gamma \omega = 0$$

and γ can be any closed curve in E of class C'. By the previous problem, since ω is a 1-form in an open set $E \subset \mathbb{R}^n$ such that

$$\int_\gamma \omega = 0$$

for any closed curve of class C', ω is exact in E.