1. True or false: if $\omega$ is a k-form and k is odd, then $\omega \wedge \omega = 0$. What if k is even and $k \geq 2$?

I claim that this statement is true. Assume that $\omega = \sum_I a_I(x)dx_I$, where $I = I_1, ..., I_n$ and each $I_u$ represents a specific $i_{u_1}, ..., i_{u_k}$ Then, we have

\[
\omega \wedge \omega = \sum_{c=1}^{n} \sum_{d=1}^{n} a_{I_c}(x)a_{I_d}(x)dx_{I_c} \wedge dx_{I_d}
\]
\[
= \sum_{c=d} a_{I_c}(x)a_{I_d}(x)dx_{I_c} \wedge dx_{I_d} + \sum_{c \neq d} a_{I_c}(x)a_{I_d}(x)dx_{I_c} \wedge dx_{I_d}
\]
\[
= \sum_{c \neq d} a_{I_c}(x)a_{I_d}(x)dx_{I_c} \wedge dx_{I_d}
\]
\[
= \sum_{c<d} a_{I_c}(x)a_{I_d}(x)dx_{I_c} \wedge dx_{I_d} + \sum_{c>d} a_{I_c}(x)a_{I_d}(x)dx_{I_c} \wedge dx_{I_d}
\]
\[
= \sum_{c<d} a_{I_c}(x)a_{I_d}(x)(dx_{I_c} \wedge dx_{I_d} + dx_{I_d} \wedge dx_{I_c})
\]

Claim: $dx_{I_c} \wedge dx_{I_d} + dx_{I_d} \wedge dx_{I_c} = 0$.

Proof of this claim: Let $I_c = i_1, ..., i_k$ and let $I_d = j_1, ..., j_k$, where k is odd.

Then $dx_{I_c} \wedge dx_{I_d} = dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k} \wedge dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_k}$

$= (-1)^k dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k} \wedge dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k} = -dx_{i_1} \wedge dx_{i_1}$.

Therefore $\omega \wedge \omega = 0$ when k is odd.

The statement is false if k is even and $k \geq 2$. Counterexample: let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$.

Then

\[
\omega \wedge \omega = (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4)
\]
\[
= dx_1 \wedge dx_2 \wedge dx_1 \wedge dx_2 + dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_4 + dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \wedge dx_3 \wedge dx_4
\]
\[
= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4
\]

And $\omega \wedge \omega \neq 0$.

2. a) Check that for any 2-simplex $\sigma = [p_0, p_1, p_2]$, we have $\partial(\partial \sigma) = 0$.

By definition, $\partial \sigma = [p_1, p_2] - [p_0, p_2] + [p_0, p_1]$.

Therefore $\partial(\partial \sigma) = [p_2] - [p_1] - ([p_2] - [p_0]) + [p_1] - [p_0] = [p_2] - [p_2] + [p_1] - [p_1] + [p_0] - [p_0] = 0$.

b) Show that for any k-chain $\Gamma$, $\partial(\partial \Gamma) = 0$.

First I will show that this is true for any k-simplex $\sigma$.

If $\sigma = [p_0, p_1, ..., p_k]$, then by definition

\[
\partial \sigma = \sum_{i=0}^{k} (-1)^i \sigma_{-i}
\]
Therefore, for any \(a, b\) such that \(0 < a < b \leq k\), since we can either remove \(p_a\) or \(p_b\) first, we are adding

\[
(-1)^a (-1)^{b-1} [p_0, p_1, \ldots, p_{a-1}, \hat{p}_a, p_{a+1}, \ldots, p_{b-1}, \hat{p}_b, p_{b+1}, \ldots, p_k] + (-1)^a (-1)^b [p_0, p_1, \ldots, p_{a-1}, \hat{p}_a, p_{a+1}, \ldots, p_{b-1}, \hat{p}_b, p_{b+1}, \ldots, p_k] = 0.
\]

Therefore, \(\partial \partial \sigma = 0\) for any \(k\)-simplex \(\sigma\). Therefore, for any \(k\)-chain \(\Gamma = \sigma_1 + \sigma_2 + \cdots + \sigma_n\),

\[
\partial(\partial \Gamma) = \partial(\partial(\sigma_1 + \cdots + \sigma_n)) = \partial(\partial \sigma_1 + \partial \sigma_2 + \cdots + \partial \sigma_n) = \partial(\partial \sigma_1) + \partial(\partial \sigma_2) + \cdots + \partial(\partial \sigma_n) = 0
\]

3. Rudin Chapter 10, #17, 18, 24, 25, 26

17. Put \(J^2 = \tau_1 + \tau_2\), where \(\tau_1 = [0, e_1, e_1 + e_2]\), \(\tau_2 = [-0, e_2, e_2 + e_1]\). Explain why it is reasonable to call \(J^2\) the positively oriented unit square in \(R^2\). Show that \(\partial J^2\) is the sum of 4 oriented affine 1-simplexes. Find these. What is \(\partial (\tau_1 - \tau_2)\)?

If we draw a diagonal line from \((0,0)\) to \((1,1)\), \(\tau_1\) is the bottom half of the unit square and \(\tau_2\) is the top half of the unit square, by definition. We want to find the orientation of these simplices. First we note that \(\tau_2 = [-0, e_2, e_2 + e_1]\) is \([0, e_2 + e_1, e_2]\]

We can write \(\tau_1(u) = 0 + A_1 u\), \(\tau_2(u) = A_2 u\) as in equation (78) on page 266, where \(u \in \mathbb{R}^2\). Then

\[
A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}
\]

and \(\det A_1 = 1\) and \(\det A_2 = 1\). By the definition on page 267 of Rudin, \(\tau_1\) and \(\tau_2\) are positively oriented. Thus \(J^2\) is positively oriented.
\[
\partial J^2 = \partial (\tau_1 + \tau_2) = \partial (\tau_1) + \partial (\tau_2) \\
= [e_1, e_1 + e_2] - [0, e_1 + e_2] + [0, e_1] + [e_2 + e_1, e_2] - [0, e_2] + [0, e_2 + e_1] \\
= [e_1, e_1 + e_2] + [0, e_1] + [e_1 + e_2, e_2] + [e_2, 0]
\]

Thus \(\partial (J^2)\) is the sum of four oriented affine 1-simplexes.

\[
\partial (\tau_1 - \tau_2) = \partial [0, e_1, e_1 + e_2] + \partial [0, e_2, e_1 + e_2] \\
= [e_1, e_1 + e_2] - [0, e_1 + e_2] + [0, e_1] + [e_2, e_1 + e_2] - [0, e_1 + e_2] + [0, e_2] \\
= [e_1, e_1 + e_2] + [0, e_1] + [e_2, e_1 + e_2] + [0, e_2] - 2[0, e_1 + e_2]
\]

18. Consider the oriented affine 3-simplex

\[\sigma_1 = [0, e_1, e_2, e_1 + e_2 + e_3]\]

in \(\mathbb{R}^3\). Show that \(\sigma_1\) (regarded as a linear transformation) has determinant 1. Thus \(\sigma_1\) is positively oriented.

Since the first element \((p_0)\) of \(\sigma_1\) is the zero vector, the transformation \(A\) associated with \(\sigma_1\) must satisfy for any \(u \in Q^3\) (the standard 3 simplex) \(\sigma_1(u) = Au\). Thus

\[A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\]

\(A\) is an upper triangular matrix, so multiplying the elements on the diagonal we get that \(\det A = 1\). Thus \(\sigma_1\) is positively oriented.

24. Let \(\omega = \sum a_i(x)dx_i\) be a 1-form of class \(C^0\) in a convex open set \(E \subset \mathbb{R}^n\). Assume \(d\omega = 0\) and prove that \(\omega\) is exact in \(E\) by completing the following outline:

Fix \(p \in E\). Define

\[f(x) = \int_{[p,x]} \]

For affine-oriented 2-simplexes \([p, x, y]\) in \(E\), since \(p, x, y\) are in \(E\) and \(E\) is convex, the line segments connecting any two of the three points are also contained entirely within \(E\). Therefore if we let \(\sigma = [p, x, y]\), the boundary of \(\sigma\) is given by \(\partial \sigma = [x, y] - [p, y] + [p, x]\). Since \(d\omega = 0\), by Stokes’ theorem

\[0 = \int_{\sigma} d\omega = \int_{\partial \sigma} \omega = \int_{[x,y]} \omega - \int_{[p,y]} \omega + \int_{[p,x]} \omega\]

Therefore

\[\int_{[p,y]} \omega - \int_{[p,x]} \omega = \int_{[x,y]} \omega\]

Note that the left hand side of this equation is equal to \(f(y) - f(x)\), and \([x, y]\) is the line segment from \(x\) to \(y\). We can parametrize this line segment by \(\gamma(t) = x + (y-x)t\). Since \(x \in \mathbb{R}^n\), we can rewrite \(\gamma(t) = (x_1 + (y_1 - x_1)t, x_2 + (y_2 - x_2)t, ..., (x_n + (y_n - x_n)t))\).
Then taking the line integral as usual, we get
\[
f(y) - f(x) = \int_{[x,y]} \omega = \sum_{i=1}^{n} \int_{0}^{1} a_i(x + (y - x)t)\gamma_i'(t)dt
= \sum_{i=1}^{n} \int_{0}^{1} a_i(x + (y - x)t)(y_i - x_i)dt
= \sum_{i=1}^{n} (y_i - x_i) \int_{0}^{1} a_i((1 - t)x + ty)dt
\]
for \(x, y \in E\). Therefore for \(x \in E\),
\[
(D_i f)(x) = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}
= \lim_{h \to 0} \frac{h \int_{0}^{1} a_i((1 - t)x + t(x + he_i))dt}{h}
= \lim_{h \to 0} \int_{0}^{1} a_i((1 - t)x + t(x + he_i))dt
= \int_{0}^{1} a_i(x)dt = a_i(x)
\]
Hence \(\omega\) is exact.

25. Assume that \(\omega\) is a 1-form in an open set \(E \subset \mathbb{R}^n\) such that
\[
\int_{\gamma} \omega = 0
\]
for every closed curve \(\gamma\) in \(E\), of class \(C'\). Prove that \(\omega\) is exact in \(E\).

Let \(\omega = \sum a_i(x)dx_i\). Want to show that there exists \(f(x)\) such that \(df = \omega\).

Fix \(p \in E\). \(E\) is open, so \(\exists \delta > 0\) such that \(N_\delta(p) \subset E\). Then for any \(x, y \in N_\delta(p)\), let \(f(x) = \int_{[p,x]} \omega\).

Let \(\gamma\) be the 1-chain \(\gamma = [x, y] - [p, y] + [p, x]\). So \(\gamma\) is a closed curve, and
\[
0 = \int_{\gamma} \omega = \int_{[x,y]-[p,y]+[p,x]} \omega = \int_{[x,y]} \omega - \int_{[p,y]} \omega + \int_{[p,x]} \omega
\]
Therefore we have
\[
f(y) - f(x) = \int_{[x,y]} \omega
\]
when \(x, y \in N_\delta(p)\). By the exact same argument as in 24 (we have the same parametrization of \([x, y]\)), \((D_i f)(x) = a_i(x)\) and \(\omega\) is exact.

26. Assume \(\omega\) is a 1-form in \(\mathbb{R}^3 - \{0\}\), of class \(C'\) and \(d\omega = 0\). Prove that \(\omega\) is exact in \(\mathbb{R}^3 - \{0\}\).

Let \(E = \mathbb{R}^3 - \{0\}\) for simplicity - note that \(E\) is open. We know that every closed continuously differentiable curve in \(E\) is the boundary of a 2-surface in \(E\). Let \(\gamma\) be a closed continuously differentiable curve in \(E\); it is the boundary of a 2-surface \(S\) in \(E\). That is, let \(S\) be the surface such that \(\partial S = \gamma\). Then by Stokes’ theorem,
\[ 0 = \int_S 0 = \int_S d\omega = \int_{\partial S} \omega \]

So
\[ \int_{\partial S} \omega = \int_\gamma \omega = 0 \]

and \( \gamma \) can be any closed curve in \( E \) of class \( C' \). By the previous problem, since \( \omega \) is a 1-form in an open set \( E \subset \mathbb{R}^n \) such that
\[ \int_\gamma \omega = 0 \]

for any closed curve of class \( C' \), \( \omega \) is exact in \( E \).