1. Consider the following region \( E = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\} \). And consider the map \( T : E \rightarrow \mathbb{R}^2 \) defined as

\[
T(x, y) = (x, (1 - x)y) = (u, v)
\]

We will want to implement the Change of Variables Theorem in this problem.

(a) (worth 2) Sketch \( T(E) \)

\[\textit{Proof.} \text{ It is } Q^2, \text{ a triangle in the first quadrant.} \]

(b) (worth 3) Show \( T \) is 1-1 in the appropriate region. State clearly what this region is that you are claiming \( T \) is 1-1 over. Also show that \( J_T(x, y) \neq 0 \) for all \( x \) in this region.

\[\textit{Proof.} \text{ The region we want to show } T \text{ is 1-1 over is open. Hence we pick}
\]

\[
E' = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}
\]

Now show \( T \) is 1-1 on \( E' \). Also,

\[
J_T(x, y) = 1 - x \neq 0 \quad \forall x \in E'.
\]

(c) (Worth 3) If we want to integrate a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) over \( E \) what conditions must it satisfy so that we can use the Change of Variable Formula as given in class.

\[\textit{Proof.} \text{ } f \text{ must be continuous on } \mathbb{R}, \text{ have compact support and } \text{supp}(f) \subseteq T(E'). \]

(d) (Worth 2) Give the Change of Variable formula.

\[\textit{Proof.} \]

\[
\int_{\mathbb{R}^2} f(u, v)dudv = \int_{\mathbb{R}^2} f(T(x, y))|J_T(x, y)|dxdy
\]