When do Prices Coordinate Markets?

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Prices are remarkable

- Markets are decentralized.
- Individuals observe prices, and buy bundles of goods that optimize their own utility functions. And somehow:
  - Markets clear! No substantial shortages or surpluses
  - The resulting allocation is pretty good.
What does the theory tell us?

• A commodity market is defined by:
  • A set of $m$ types of discrete goods $g \in G$, each with supply $s_g \geq 1$
  • A set of $n$ buyers $i \in N$, each with a valuation function $v_i: 2^G \rightarrow [0,1]$

• A feasible allocation is a function $\mu: N \rightarrow 2^G$ such that:
  • For each $g \in G$: $|\{i \in N: g \in \mu(i)\}| \leq s_g$

• The optimal welfare in a market is:

$$OPT = \max_{\mu} \sum_{i=1}^{n} v_i(\mu(i))$$
What does the theory tell us?

• When facing prices, people are self interested.
  • The demand set $D_i(p)$ for a bidder $i$ at prices $p$ is the set of bundles that he values maximally, given the prices:
    $$D_i(p) = \arg \max_{S \subseteq G} \left( v_i(S) - \sum_{g \in S} p_g \right)$$
  • Buyers will always choose to buy a bundle in their demand set.
What does the theory tell us?

- A set of prices $p = (p_1, ..., p_m)$ are Walrasian equilibrium prices if there exists a feasible allocation $\mu$ such that:
  - For every buyer $i \in N$, $\mu(i) \in D_i(p)$, and
  - $|\{i : g \in \mu(i)\}| < s_g$ only if $p_g = 0$
- For such a $\mu$, we say $(p, \mu)$ form a Walrasian equilibrium.
What does the theory tell us?

• Some remarkable facts:
  • If \((p, \mu)\) is a Walrasian equilibrium, then \(\sum_i v_i(\mu(i)) = OPT\) (First Welfare Theorem)
  • If buyer valuations satisfy the \textit{gross substitutes property}, then Walrasian equilibria are guaranteed to exist! [KC82,GS99]
  • In fact, natural ascending price auction dynamics converge to them!
  • In fact, many such prices – the set of Walrasian equilibrium prices forms a lattice. The \textit{minimal} Walrasian equilibrium prices are focal: many tatonnemont processes converge to them, and they correspond to VCG prices for unit-demand valuations.

Interpretation?
  “Prices \textit{optimally} coordinate a large class of markets.”
Some problems with that interpretation

• No coordination?
  • Each bidder $i$ may have many bundles in his demand set $D_i(p)$, and to obtain the optimal allocation $\mu$, these ties have to be broken in a coordinated manner.
The coordination problem

Over-Demand = n-1
The coordination problem

But that example is non-generic!
The coordination problem

• There will always be over-demand at the minimal Walrasian equilibrium prices!
  • Suppose otherwise: $p(g) > 0$ and $OD(g) = 0$
    • Must have $|\{i \in N : g \in \mu(i)\}| = s_g$
    • All buyers $j$ such that $g \notin \mu(j)$ cannot demand any bundle with $g$, otherwise $OD(g) > 0$.
    • All the other buyers strictly prefer a bundle that does not contain $g$.
    • Then we could take any nonzero price and lower it by $\epsilon$
    • The prices would remain Walrasian, contradicting minimality.

The best we can hope for is approximation.
Some problems with that interpretation

• Where do the prices come from?
  • There are many natural interactive auction procedures that converge to Walrasian prices.
  • But in practice, we face fixed prices and do not engage in auctions.
Where do prices come from?

• In practice, prices encode “distributional information”.

• But:
  • How much distributional information is necessary?
  • Implies we are facing approximate equilibrium prices – only exacerbates coordination.
This work

$1.25 \quad $5.99 \quad $0.50\quad \text{Minimal equilibrium prices}

\begin{align*}
m \text{ types of goods } g \text{ with supply } s_g \\
n \text{ buyers sampled iid from } \Pi
\end{align*}

Distribution \( \Pi \) over valuation functions
This work

$1.25 \quad $5.99 \quad $0.50$

$m$ types of goods $g$ with supply $s_g$

 Buyers buy most preferred bundles, break ties arbitrarily

$n$ buyers sampled iid from $\Pi$

Distribution $\Pi$ over valuation functions
This work

• How much over-demand do we expect? How high is welfare?

• What do these things depend on?
  • Particulars of the distribution $\Pi$?
  • Complexity of valuation functions $v_i$?
  • Number of buyers $n$?
  • Number of goods $m$?
This work

• First question: When do the \textit{exact} minimal equilibrium prices induce little over-demand?
  • Remember: Not always. And we must always have some over-demand.

• Second question: How well do equilibrium prices computed on a sample generalize to new buyers drawn from the same distribution?
  • And how much data is needed?
First Question

• Warmup: Unit demand buyers
  • \( v_i(S) = \max_{g \in S} v_i(\{g\}) \equiv \max_{g \in S} v_{i,g} \)
  • (i.e. buyers just want 1 item)

• Genericity Assumption:
  \[ \sum_{i,g} a_{i,g} v_{i,g} = 0 \text{ with } a_{i,g} \in \{-1,0,1\} \text{ iff } a_{i,g} = 0 \text{ for all } i, g. \]
  (i.e. valuations are linearly independent over \{-1,0,1\})

Notes on assumption:

• Implies the welfare-optimal allocation \( \mu \) is unique.

• Satisfied with probability 1 for any continuous perturbation of valuations.
Over-Demand

• Theorem: If valuations satisfy our genericity assumption, and \( p \) are the minimal Walrasian equilibrium prices, then for every good \( g \):

\[
|\{i \in N: g \in D_i(p)\}| \leq s_g + 1
\]

(no matter how people break ties, over-demand on any good is at most 1)
Proof

• Fix the optimal allocation $\mu$, minimal equilibrium prices $p$.

• Construct a graph $G = (V, E)$ where:
  • $V = \{1, \ldots, m\}$ – vertices are types of goods.
  • $(g, g') \in E$ for every buyer $i$ with $\mu(i) = g$, $g' \in D_i(p)$ and $g' \neq g$
Claim 1: The graph must be acyclic.
Proof: Otherwise players could swap allocations around a cycle, and arrive at a distinct max-welfare allocation $\mu'$. 
Proof

So, we can topologically sort the graph, rename vertices in order. First vertex has indegree zero.
Claim 2: Any good with in-degree zero must have price 0
Proof: Otherwise we could lower the price, contradicting minimality.
Claim 3: All prices $p_g$ can be written as linear combinations of valuations $v_{i,h}$ where $h \leq g$ and coefficients are in $\{-1,0,1\}$.
Proof: Base case: $p_1 = 0$
Proof

Claim 3: All prices $p_g$ can be written as linear combinations of valuations $v_{i,h}$ where $h \leq g$ and coefficients are in $\{-1,0,1\}$.

Proof: Inductive case: If $g$ has positive in-degree, there is a buyer $i$ with $\mu(i) = g'$ for $g' < g$, and $g \in D_i(p)$. i.e:

$$v_{i,g'} - p_{g'} = v_{i,g} - p_g \text{ or:}$$

$$p_g = v_{i,g} - v_{i,g'} + p_{g'}$$
Proof

Claim 4: All goods $g$ have in degree $\leq 1$
Proof

Claim 4: All goods $g$ have in degree $\leq 1$

Proof: Suppose not. Then there are two buyers $i \neq i'$ with:

$v_{i,\mu(i)} - p_{\mu(i)} = v_{i,g} - p_g$ and $v_{i',\mu(i')} - p_{\mu(i')} = v_{i',g} - p_g$

This gives us two expressions for $p_g$. Subtracting them:

$v_{i,g} - v_{i',g} + v_{i,\mu(i)} - v_{i',\mu(i')} + p_{\mu(i)} - p_{\mu(i')} = 0$

The coefficients aren’t all zero since $\mu(i), \mu(i') < g$

Contradicts genericity!
Welfare

• We have shown that over-demand is low when buyers grab any good from their demand set.

• Impose the following rule: If a buyer is indifferent to empty allocation or getting a good, then she takes a good.

• If buyers grab demanded goods \( \{b_1, \ldots, b_n\} \) following the rule above, then the resulting welfare is close to optimal:
  - \( \text{Welfare}(\{b_1, \ldots, b_n\}) \geq \text{OPT} - 2m \)
Extending Result

Over-Demand and Welfare Results not limited to unit demand valuations!

We now consider buyers getting bundles of goods.
Gross Substitutes Prelims

• A valuation \( v: 2^G \rightarrow [0,1] \) obeys GS if for all price vectors \( p' \geq p \), and \( S \in D(p) \) there exists a bundle \( S' \in D(p') \) such that
  - \( S \cap \{g: p(g) = p'(g)\} \subseteq S' \)

• It is known that for GS valuation \( v \), the collection of **minimal demand sets**
  \[ D^*(p) = \{ S \in D(p): S' \notin D(p), \forall S' \subset S \} \]
forms the bases of a matroid.
Exchange Property for Matroids

$B^{(1)}$ $B^{(2)}$

$g_1$ $g_6$
$g_2$ $g_4$
$g_3$ $g_5$
$g_7$
$g_8$
Exchange Property for Matroids

$B^{(1)} \cup g_7 \setminus g_1 \in D_i^*(p)$

Diagram:
- Set $B^{(1)}$ with elements $g_2, g_3, g_4, g_5$.
- Set $B^{(2)}$ with elements $g_6, g_8$.
- An element $g_7$ is exchanged from $B^{(1)}$ to $B^{(2)}$. 
Exchange Property for Matroids

\[ B^{(1)} \cup g_6 \setminus g_2 \in D_i^*(p) \quad \text{and} \quad B^{(1)} \cup g_8 \setminus g_2 \in D_i^*(p) \]
Exchange Property for Matroids

\[ B^{(1)} \cup g_7 \setminus g_3 \in D_i^*(p) \quad B^{(1)} \cup g_8 \setminus g_3 \in D_i^*(p) \]
Swap Graph for GS

- Let \( \{v_i\} \) be GS valuations for all \( n \) buyers. Fix Walrasian equilibrium \((p, \mu)\) with minimal prices, and minimum demand sets \( M_1, \ldots, M_n \) where \( M_i \subseteq \mu(i) \).

- Have a node for every good in \( G \)

- There is an edge \((a, b)\) for every buyer \( i \) where \( a \in M_i \) and \( b \notin \mu(i) \) and there exists a \( B \in D_i^*(p) \) with \( b \in B \) and

\[
M_i \cup b \setminus a \in D_i^*(p)
\]
Swap Graph for GS

• What if buyers are indifferent to bundles of different sizes?
• Include a null node ⊥
• If \( p(b) > 0 \), there is an edge from \( \perp \) to \( b \) for each buyer \( i \) that has minimum demand set \( B_i \in D_i^*(p) \) and \( B_i \cup b \in D_i(p) \setminus D_i^*(p) \)
Proof Outline for Over-Demand

• Define a genericity condition for GS valuations
• Show the swap graph is acyclic
• Show source nodes in the swap graph have price zero
• Show that the prices of goods can be written as an integer linear combination of “weights” from previous goods in a topological sort of the nodes.
• Bound in-degree.
Matroid Based Valuations

• Because the definition of Gross Substitutes is axiomatic rather than constructive, it is not clear if any GS valuation satisfies some generic condition.

• **Matroid Based Valuations** gives a constructive way to define valuations that are contained in GS.

• Conjectured by [Ostrovsky, Paes-Leme 15] that $MBV$ is equal to $GS$. 
Matroid Based Valuations

• A valuation $v$ is in VIWM if there exists a matroid $M = (I, G)$ and weights $\{w_g\}$ such that
  $v(S) = \max_{T \subseteq S, T \in I} \sum_{g \in T} w_g$

• Endowment operation: $v(S) = v'(S \cup J) - v'(J)$ where $T \cap J = \emptyset$.

• Merge operation: $v(S) = \max_{(S_1, S_2) = s} v_1(S_1) + v_2(S_2)$.

• MBV is the smallest class of valuations that contain VIWM and is closed under endowment and merge operations.
Generic MBV

• The collection of buyers valuations \( \{v_i\} \) are GMBV if they are MBV and all weights \( W \) for all the buyers and all the goods are linearly independent over the integers

\[
\sum_{w \in W} \alpha_w w = 0 \quad \text{for} \quad \{\alpha_w\} \in \mathbb{Z} \\
\text{iff} \quad \alpha_w = 0 \quad \text{for every} \quad w \in W
\]
Over-Demand for GMBV

• We can show that for buyers with GMBVs that take only minimum demand bundles at the minimal Walrasian prices, then the over-demand for any good is at most 1 – ignores the use of the null node and is bad for bounding welfare.

• The non-degenerate correspondence for buyer $i$ is,
  • $\widehat{D}_i(p) = \{S \in D_i(p): v_i(S \setminus g) < v_i(S), \forall g \in S\}$

• If buyers take bundles in their non-degenerate correspondence, then over-demand is at most 1.
Welfare for GMBV

• We bound the over-demand for buyers that take any non-degenerate bundle by 1
• If buyers take max cardinality non-degenerate bundles, \( \{B_1, \ldots, B_n\} \), then the Welfare is close to optimal:

\[
Welfare(B_1, \ldots, B_n) \geq OPT - 2m
\]
Second Question

• We have a condition under which (exact) equilibrium prices computed on a population induce low over-demand.

• How well does this generalize if we use the same prices on a new population?
Over-Demand Generalization

**Punchline:**
On a fresh sample of buyers, the demand for any good $g$ satisfies:

$$\left| |\{i : g \in D_i(p)\}| - s_g \right| \leq O\left(\sqrt{s_g \cdot m \cdot \log \frac{m}{\delta}}\right)$$

with probability $(1 - \delta)$.

i.e. the total demand for any good is w.h.p. within a $(1 + \epsilon)$ factor of the supply whenever:

$$s_g \geq \tilde{O}\left(\frac{m}{\epsilon^2}\right)$$
Proof Outline for Generalization

• Fix a tie breaking rule agents $i$ use to select bundles $S_{v_i}(p) \in D_{v_i}(p)$ given prices $p$.

• For a fixed $p \in \mathbb{R}^m$ define $f_p(v_i) = S_{v_i}(p)$ and for each $g \in G$ define $d_p^g(v_i) = \begin{cases} 1 & \text{if } g \in S_{v_i}(p) \\ 0 & \text{otherwise} \end{cases}$

• Given a distribution $\Pi$ over valuation functions, the expected demand for $g$ given prices $p$ is:

$$n \cdot \mathbb{E}_{v_i \sim \Pi} \left[ d_p^g(v_i) \right]$$
Uniform Convergence

• It suffices to obtain uniform convergence of the empirical averages over the sets of functions:
  \[ C^g = \{ d^g_p : p \in \mathbb{R}^m \} \]
  to their expectations.

  (then, with high probability over the draws of two samples of bidders, for every price vector, demand is similar on both samples.)

  (In particular, for the Walrasian prices computed on the first sample, for which we know over-demand is small)
Learning and Uniform Convergence

• For one dimensional, Boolean learning problems, learning over $C^g \iff$ Uniform convergence over $C^g$.
  • But not so for multi-dimensional/real valued learning problems.

{\forall_p:V \rightarrow 2^g}$ is learnable with $O\left(\frac{m}{\varepsilon}\right)$-samples.

$C^g = \{a_p^g:V \rightarrow \{0, 1\}\}$ is learnable with $O\left(\frac{m}{\varepsilon}\right)$-samples.

Uniform convergence over $C^g$ with $O\left(\frac{m}{\varepsilon^2}\right)$ samples.
Welfare Generalization

**Punchline:**
The welfare induced by the chosen price vector on a new sample of buyers is with high probability at least:

$$(1 - \epsilon) \cdot OPT$$

Whenever $OPT \geq \tilde{O} \left( \frac{m^4 \sqrt{n}}{\epsilon^2} \right)$
Outline of Proof

• Welfare is a real valued function
• Directly bound the Pseudo-Dimension via shattering.
• Want to bound the following: For $n$ buyers, how many distinct allocations can be induced by varying over all price vectors?
• We can bound this by $2^{O(m^2)}$, so that Pseudo-Dimension is no more than $\tilde{O}(m^2)$. 
Do prices coordinate markets?

Generically, they do!

THANKS