NOTES ON DEFORMATION QUANTIZATION

SHILIN YU

Abstract. ... 

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1. Deformation theory

1.1. Deformation of associative algebras. Fix a field \( k \) of characteristic zero. Let \( A \) be an associative algebra. A formal deformation of \( A \) is an associative \( k \)-algebra structure on \( A \) such that \( A \approx A/J/\hbar K \) as algebras. We denote the multiplication on \( A/J/\hbar K \) by the star product \( \star : A/J/\hbar K \times A/J/\hbar K \to A/J/\hbar K \), then it is determined by its values on the subspace \( A \subset A/J/\hbar K \). We write for any \( f, g \in A \),

\[
f \star g = fg + B_1(f, g)\hbar + B_2(f, g)\hbar^2 + \cdots + B_n(f, g)\hbar^n + \cdots. \tag{1}
\]

where \( B_n : A \otimes A \to A \).

Let \( J \) be the group of \( k \)-module automorphisms \( E \) of \( A/J/\hbar K \), such that

\[
E(f) \equiv f \mod tA/J/\hbar K,
\]
i.e., \( E \) is of the form

\[
E(f) = f + E_1(f)\hbar + E_2(f)\hbar^2 + \cdots, \quad E_i \in \text{Hom}_k(A, A).
\]

Key words and phrases. ...
Then $J$ acts on the set of all formal deformations. Any $E \in J$ transforms a star product $\ast$ to $\ast'$ via

$$f \ast' g = E(E^{-1}(f) \ast E^{-1}(g)).$$

(2)

In this case, we say that star and $\ast'$ are gauge equivalent. We want to classify formal deformations of $A$ up to gauge equivalences.

Let us first look at the first-order deformations of $A$, i.e., $\mathbb{k}[[h]]$-algebra structures on $A[[h]]/(h) = A \oplus h \cdot A$. If we expand the associativity condition $(f \ast g) \ast h = f \ast (g \ast h)$ according to (1) and compare the coefficient of the first-order term, we get

$$B_1(fg,h) + B_1(f,g)h = fB_1(g,h) + B_1(f,gh),$$

(3)

or equivalently,

$$fB_1(g,h) - B_1(fg,h) + B_1(f,gh) - B_1(f,g)h = 0.$$  

(4)

This means that $B_1 : A \otimes A \to A$ is a Hochschild 2-cocycle. Now suppose there is a gauge transform $E = \text{Id}_A + hE_1$ on $B_1$, which maps a star product $\ast$ with first coefficient $B_1$ to $\ast'$ with first coefficient $B'_1$. Then we have

$$B'_1(f,g) = B_1(f,g) - fE_1(g) + E_1(fg) - E_1(f)g.$$  

(5)

Hence $B'_1$ differs from $B_1$ with a Hochschild 2-coboundary.

**Theorem 1.1.** The set of gauge equivalence classes of first-order deformation of an associative algebra $A$ is bijective to the second Hochschild cohomology $HH^2(A,A)$.

1.2. Hochschild complex and dg-Lie algebra. Let $A$ be as before. The Hochschild cochain complex $C^\bullet(A,A)$ is defined to be

$$C^n(A,A) := \text{Hom}_k(A^\otimes_n, A)$$

with the differential $b : C^n(A,A) \to C^{n+1}(A,A)$ given by

$$b\phi(a_1, a_2, \cdots, a_{n+1}) = a_1 \phi(a_2, \cdots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \phi(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) + (-1)^{n+1} \phi(a_1, \cdots, a_n a_{n+1},$$

for any $\phi \in C^n(A,A)$. The cohomology of $(C^\bullet(A,A),b)$ is the Hochschild cohomology $HH^\bullet(A,A)$ of the algebra $A$.

Now suppose we have a formal deformation $(A[[h]], \ast)$ of $A$. Denote by $\mu : A \otimes A \to A$ the original multiplication of $A$ and by $\mu_h : A[[h]] \otimes_k A[[h]] \to A[[h]]$ the star product. We will
rewrite the associativity condition of $\mu_h$ in terms of a DGLA structure on the Hochschild complex $(C^*(A, A), b)$. Let $\phi$ be a Hochschild $p$-cochain and $\varphi$ be a Hochschild $q$-cochain. The Gerstenhaber product of $\phi$ and $\varphi$ is the $(p + q - 1)$-cochain defined by

$$\phi \circ \varphi(a_1, \cdots, a_{p+q-1}) = \sum_{i=0}^{p-1} (-1)^i (q+1) \phi(a_1, \cdots, a_i, \varphi(a_{i+1}, \cdots, a_{i+q}), a_{i+q+1}, \cdots, a_{p+q-1}).$$

The Gerstenhaber bracket is then defined by

$$[\phi, \varphi] = \phi \circ \varphi - (-1)^{(p-1)(q-1)} \varphi \circ \phi$$

**Definition 1.2.** A differential graded Lie algebra (DGLA) is a complex $(L^\bullet, d)$ endowed with a Lie bracket $[\ , \ ] : L^\bullet \otimes L^\bullet \to L^\bullet$ of degree 0 which is

- antisymmetric: $[x, y] = (-1)^{|x||y|} [y, x]$,
- Jacobi identity: $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|} [y, [x, z]]$,
- compatibility: $d[x, y] = [dx, y] + (-1)^{|x||y|} [x, dy]$.

**Lemma 1.3.** The shifted Hochschild complex $C^*(A, A)[1] = C^{*+1}(A, A)$ with the Gerstenhaber bracket is a DGLA.

Note that the differential $b$ can be rewritten in terms of the Gerstenhaber bracket and the multiplication $\mu$:

$$d\phi = \pm [\mu, \phi]$$

We can form $C^\bullet(A, A)[[h]]$, the Hochschild complex depending formally on $h$, and extend the differential $b$ and the Gerstenhaber bracket into a $k[[h]]$-linear one on the new complex, so that it is again a DGLA. Regard $\mu_h$ as an element in $C^2(A, A)[[h]]$, then we have

$$[\mu_h, \mu_h](f, g, h) = 2(\mu_h(\mu_h(f, g), h) - \mu_h(f, \mu_h(g, h))) = 0$$

since $\mu_h$ is associative. Thus we have

$$[\mu_h, \mu_h] = 0 \in C^3(A, A)[[h]].$$

(6)
If we write
\[ \mu_h(f,g) = fg + B(f,g), \quad \text{or} \quad \mu_h = \mu + B, \quad B \in C^2(A,A) \otimes \mathfrak{m}, \]
where \( \mathfrak{m} = \hbar k[[\hbar]] \) is the maximal ideal of \( k[[\hbar]] \), then (6) can be rewritten as the *Maurer-Cartan equation*
\[ bB + \frac{1}{2}[B,B] = 0, \quad (7) \]
and we say that \( B \in C^1(A,A)[1] \otimes \mathfrak{m} \) is a *Maurer-Cartan element* of the DGLA \( L_A^\bullet = C^\bullet(A,A)[1] \otimes \mathfrak{m} \) (which is a subDGLA of \( C^\bullet(A,A)[1] \otimes k[[\hbar]] \)). On the other hand, notice that the gauge action (2) on the set of \( \mu_h \)'s is the exponential of the action of the pronilpotent Lie algebra \( L_A^0 = C^1(A,A) \otimes (\hbar) \) given by
\[ \rho_X(\mu_h) = [X,\mu + B] = [X,B] - bX \in L_A^1, \quad X \in L_A^0. \quad (8) \]
Thus we arrive at the following general definition.

**Definition 1.4.** Let \( (L^\bullet,d) \) be a (nilpotent/pronilpotent) DGLA. The set of all Maurer-Cartan elements of \( L^\bullet \) is defined to be
\[ MC(L^\bullet) = \{ x \in L^1|dx + \frac{1}{2}[x,x] = 0 \}. \]
The gauge action of \( \exp L^0 \) on \( MC(L^\bullet) \) is the affine action on \( L^1 \) whose differential is the Lie algebra homomorphism
\[ L^0 \rightarrow \text{Lie(Aff}(L^1)), \quad X \mapsto (x \mapsto [X,x] - dX). \]
The set of formal deformations governed by \( L^\bullet \) over the formal disk \( \text{Spf} k[[\hbar]] \) is
\[ \text{Def}_L(k[[\hbar]]) := MC(L^\bullet \otimes \mathfrak{m})/ \exp(L^0 \otimes \mathfrak{m}). \]
Note that if \( f : L^1_1 \rightarrow L^1_2 \) is a homomorphism of DGLAs, then it induces map \( \text{Def}_{L_1} \rightarrow \text{Def}_{L_2} \).

1.3. **Deformation quantization of Poisson manifolds.** Our primary interest is in the deformation of the algebra of smooth functions \( A = C^\infty(M) \) (which is commutative!) on a smooth manifold with extra structures. For this purpose, the usual Hochschild complex \( C^\bullet(A,A) \) is too big. Instead, we use a subcomplex \( D^\bullet_{\text{poly}} \) of the shifted Hochschild complex \( C^\bullet(A,A)[1] \), which consists of those cochains that are polydifferential operators. For
instance, a cochain in $D^n_{poly}$ under some local coordinates $(x_i)$ is of the form

$$f_0 \otimes \cdots \otimes f_n \mapsto \sum_{I_0, \cdots, I_n} C^{I_0, \cdots, I_n}(x) \cdot \partial_{I_0}(f_0) \cdots \partial_{I_n}(f_n).$$

The corresponding cohomology groups will be referred as the Hochschild cohomology $HH^\bullet(A, A)$.

This means that in the expansion of the star product (1), we require $B_n$ to be polydifferential operators. Suppose there is a formal deformation of $C^\infty(M)$ with a star product $\star$, we define a Poisson bracket on $C^\infty(M)$ by

$$\{f, g\} = \frac{f \star g - g \star f}{2\hbar} = \frac{1}{2}(B_1(f, g) - B_1(g, f)) = B_1^-(f, g), \quad \forall f, g \in C^\infty(M).$$

where $B_1^-(f, g)$ is the antisymmetric part of $B_1$. Notice that the gauge action (5) only changes the symmetric part of $B_1$. In fact, the symmetric part of $B_1$ can always be eliminated by a gauge transform. So from now on let us assume $B_1$ is antisymmetric. The Poisson bracket acts as derivations in both parameters, i.e.,

$$\{f, gh\} = \{f, g\}h + g\{f, h\},$$

and it satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

To see this, we take the commutators of the star product

$$[f, g] = f \star g - g \star f$$

$$= \hbar(B_1(f, g) - B_1(g, f)) + \hbar^2[B_2(f, g) - B_2(g, f)] + O(\hbar^3)$$

$$= \hbar^2B_1(f, g) + \hbar^2[B_2(f, g) - B_2(g, f)] + O(\hbar^3),$$

so

$$[f, g] \star h = \hbar^2B_1(f, g)h + \hbar^2((B_2(f, g) - B_2(g, f))h + 2B_1(B_1(f, g), h)) + O(\hbar^3),$$

and

$$h \star [f, g] = \hbar^2hB_1(f, g) + \hbar^2(h(B_2(f, g) - B_2(g, f)) + 2B_1(h, B_1(f, g))) + O(\hbar^3).$$

Hence

$$[[f, g], h] = \hbar^24B_1(B_1(f, g), h) + O(\hbar^3) = \hbar^24\{f, g\}, h\} + O(\hbar^3),$$

and the Jacobi identity for the Poisson bracket comes from the $\hbar^2$ term of the Jacobi identity for the commutator of $\star$. We rephrase the Poisson bracket as a bivector field $\alpha \in \Gamma(\wedge^2TM)$,
such that \( \langle \alpha, df \otimes dg \rangle = B^{-1}(f, g) = \{f, g\} \). Then the Jacobi identity can be written as 
\[ [\alpha, \alpha]_{SN} = 0. \]
The bracket \([,]_{SN}\) here is the Schouten-Nijenhuis bracket which will be introduced in due course.

In other words, we have bijections between the set of Poisson structures, the second Hochschild cohomology \( HH^2(A, A) \) and the gauge equivalence classes of all first-order deformations.

**Questions 1.5.** Given a Poisson structure on \( M \), can it always be lifted to a formal deformation of \( C^\infty(M) \)?

**Example 1.6 (The Moyal-Weyl quantization).** Let us start with the special case when the manifold is \( V = \mathbb{R}^2 = \{(p, q)\} \) equipped with the symplectic form \( \omega = 2dp \wedge dq \). The corresponding Poisson bivector is \( \alpha = 2\partial_p \wedge \partial_q \). The poisson bracket on \( C^\infty(V) \) is given by

\[
\{f, g\} = \frac{\partial f}{\partial x_p} \frac{\partial g}{\partial x_q} - \frac{\partial f}{\partial x_q} \frac{\partial g}{\partial x_p}.
\]

A relevant noncommutative algebra here is the Weyl algebra

\[ A_\hbar = \mathbb{R}(p, q)\mathbb{[}\hbar]/\langle pq - qp - 2\hbar \rangle. \]
We show that it is a deformation of the algebra \( \mathbb{R}[V] = S(V^*) \) of polynomial functions over \( V \). There is a \( \mathbb{R}[\hbar] \)-linear symmetrization map

\[ Sym : \mathbb{R}[V]\mathbb{[}\hbar] \rightarrow A_\hbar \]
given by

\[ Sym(v_1v_2 \cdots v_n) = \frac{1}{n!} \sum_\sigma [v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}] \in A_\hbar. \]
By some filtration argument one can see that this is a isomorphism of \( \mathbb{R}[\hbar] \)-modules. Thus we can define a star product on \( \mathbb{R}[V] \) by

\[ f \star g = Sym^{-1}(Sym(f) \cdot Sym(g)). \]

On the other hand, as a subalgebra of \( C^\infty(V) \), \( \mathbb{R}[V] \) is invariant under the Poisson bracket. We show that the first-order term of the expansion of the star product on \( \mathbb{R}[V] \) is equal to the Poisson bracket, i.e.,

\[
\left. \frac{f \star g - g \star f}{2\hbar} \right|_{\hbar=0} = \{f, g\}, \quad \forall f, g \in \mathbb{R}[V]. \quad (9)
\]
Notice that both sides of (9) are derivations with respect to \( f, g \) and the usual commutative product of functions. Thus we only need to check the special case when \( f(p, q) = p \) and \( g(p, q) = q \). Indeed, in \( A_\hbar \) we have

\[
\text{Sym}(p \star q - q \star p) = p \otimes q - q \otimes p = 2\hbar,
\]

thus

\[
p \star q - q \star p = 2\hbar = 2\hbar\{p, q\}.
\]

One can show that the coefficients \( B_n \) in the expansion of the star product are differential operators in \( f \) and \( g \), so we can use this expansion to extend the star product to \( C^\infty(V) \), which is the Moyal-Weyl product. It is given by the formula

\[
f \star g = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left( \frac{\partial}{\partial p'} \frac{\partial}{\partial q'} - \frac{\partial}{\partial q'} \frac{\partial}{\partial p'} \right)^n f(p', q') g(p'', q'') \bigg|_{p' = p'' = p, q' = q'' = q}
\]

where \( \mu \) is the usual commutative product of functions. To see it, let us consider the star product of \( f(p, q) = e^{ap + bq} \) and \( g(p, q) = e^{cp + dq} \), where \( a, b, c \) and \( d \) are arbitrary real numbers. We write \( x = ap + bq \) and \( y = cp + dq \). Then the commutator \( \{x, y\} = 2\hbar(ad - bc) \) (with respect to \( \star \)) is a central element in \( C^\infty(V) \) and all higher iterations of the commutator vanish. By Campbell-Baker-Hausdorff theorem, we have

\[
e^x \star e^y = e^{x+y + \frac{1}{2}\{x, y\}} = e^{\frac{1}{2}\{x, y\}} e^{x+y} = e^{\hbar(ad - bc)} fg.
\]

Hence

\[
f \star g = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} (ad - bc)^n fg
\]

More generally, let \( M = \mathbb{R}^d \) and \( \alpha = \sum_{i,j} \alpha^{i,j} \partial_i \wedge \partial_j \) is a Poisson bivector with constant coefficients, where \( \alpha^{i,j} = -\alpha^{j,i} \). The corresponding Poisson bracket is given by

\[
\{f, g\} = \langle \alpha, df \otimes dg \rangle = \sum_{i,j} \alpha^{i,j} \partial_i f \partial_j g.
\]
The Moyal-Weyl product is given by the formula

\[ f \ast g = fg + \hbar \sum_{i,j} \alpha^{i,j} \partial_i(f) \partial_j(g) + \frac{\hbar^2}{2} \alpha^{i,j} \alpha^{k,l} \partial_i \partial_k(f) \partial_j \partial_l(g) + \cdots \]

\[ = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{i_1, \ldots, i_n} \prod_{k=1}^{n} \alpha^{i_k j_k} \left( \prod_{k=1}^{n} \partial_{i_k} \right)(f) \cdot \left( \prod_{k=1}^{n} \partial_{j_k} \right)(g) \]

\[ = \mu \circ e^{\hbar \alpha} \circ (f \otimes g). \]

2. Kontsevich’s formality theorem

We have seen that \( H^1(D^\bullet_{\text{poly}}) = HH^2(A,A) \) is isomorphic to \( \wedge^2 TM \). In fact, there is a more general result due to Hochschild-Kostant-Rosenberg, of which a variation used by Kontsevich is as follows. Let

\[ T^m_{\text{poly}} = \Gamma(M, \wedge^{n+1} TM), \quad n \geq -1. \]

Then we can define a map of cochain complexes

\[ I_{\text{HKR}} : (T^\bullet_{\text{poly}}, 0) \to (D^\bullet_{\text{poly}}, b) \]

by

\[ I_{\text{HKR}}(\xi_0 \wedge \cdots \wedge \xi_n) = \left( f_0 \otimes \cdots \otimes f_n \mapsto \frac{1}{(n+1)!} \sum_{\sigma} \text{sgn}(\sigma) \prod_{i=0}^{n} \xi_{\sigma(i)}(f_i) \right). \]

**Theorem 2.1 (HKR).** Then \( I_{\text{HKR}} \) is a quasi-isomorphism between complexes.

Since \( D^\bullet_{\text{poly}} \) is a DGLA, its cohomology group \( T^\bullet_{\text{poly}} \) admits a graded Lie algebra structure, which is given by the Schouten-Nijenhuis bracket:

\[ [\xi_0 \wedge \cdots \wedge \xi_k, \eta_0 \wedge \cdots \wedge \eta_l]_{SN} = \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{i+j+k+1} [\xi_i, \eta_j] \wedge \xi_0 \wedge \cdots \wedge \xi_k \wedge \eta_0 \wedge \cdots \wedge \eta_l. \]

for \( k, l \geq 0 \). For \( k \geq 0 \),

\[ [\xi_0 \wedge \cdots \wedge \xi_k, h]_{SN} = \sum_{i=0}^{k} (-1)^i \xi_i(h) \cdot \xi_0 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_k. \]

It is tempting to say that \( I_{\text{HKR}} \) is a homomorphism of DGLAs, then we can map a Maurer-Cartan element of \( T^\bullet_{\text{poly}} \hat{\otimes} \mathfrak{m} \), which is a Poisson bivector \( \alpha \), to a Maurer-Cartan element of \( D^\bullet_{\text{poly}} \hat{\otimes} \mathfrak{m} \), which is a formal deformation of \( C^\infty(M) \), and hence prove answer the Question 1.5! Unfortunately it is not the case... but Kontsevich showed that \( I_{\text{HKR}} \) can be
fixed by an $L_{\infty}$-morphism, which gives bijection between deformations arising from the two DGLAs.

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA