Math 104 – Calculus

6.4 Surface Area
Surface Area

• To find the surface area of a solid of revolution, we take a partition and approximate by strips/bands.
Riemann Sum Approximation

- The surface area of a band can be calculated in terms of its radius (distance to the axis of rotation) and its arc length.

\[
\sum_{k=1}^{n} 2\pi \cdot f(x_k) \sqrt{1 + f'(x_k)^2} \Delta x_k \xrightarrow{\Delta x_k \to 0} \int_{a}^{b} 2\pi f(x) \sqrt{1 + f'(x)^2} \, dx
\]
**Surface Area Formula**

**DEFINITION** If the function \( f(x) \geq 0 \) is continuously differentiable on \([a, b]\), the area of the surface generated by revolving the graph of \( y = f(x) \) about the \( x \)-axis is

\[
S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx. \tag{3}
\]

**Surface Area for Revolution About the \( y \)-Axis**

If \( x = g(y) \geq 0 \) is continuously differentiable on \([c, d]\), the area of the surface generated by revolving the graph of \( x = g(y) \) about the \( y \)-axis is

\[
S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} \, dy. \tag{4}
\]
### Rules

In general the surface area formula is \( \int_a^b 2\pi r ds \).

<table>
<thead>
<tr>
<th></th>
<th>( y = f(x) )</th>
<th>( x = g(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>x-axis</strong></td>
<td>( r = f(x) )</td>
<td>( r = y )</td>
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<tr>
<td></td>
<td>( ds = \sqrt{1 + f'(x)^2} , dx )</td>
<td>( ds = \sqrt{1 + g'(y)^2} , dy )</td>
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<tr>
<td><strong>y-axis</strong></td>
<td>( r = x )</td>
<td>( r = g(y) )</td>
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</table>

• Note: when rotate about a general line, add appropriate constant term in the radius \( r \).
Examples

1. Find the area of the surface formed by revolving the curve $y = \sqrt{x}$ on the interval $0 \leq x \leq 2$ about the $x$-axis

2. Find the area of the surface formed by revolving the curve $y = x^2$ on the interval $0 \leq x \leq \sqrt{2}$ about the $y$-axis.
Examples

3. Find the surface area of sphere of radius $a$. 