1.2 Define \( f(t) = \mathbf{x}(t) \cdot \mathbf{V} \) then
\[
f'(t) = \mathbf{x}'(t) \cdot \mathbf{V} + \mathbf{x}(t) \cdot \mathbf{V}'
\]
\[
= \mathbf{x}'(t) \cdot \mathbf{V} \quad \text{since } \mathbf{V} \text{ is a constant vector}
\]
\[
= 0 \quad \text{since } \mathbf{x}'(t) \text{ is orthogonal to } \mathbf{V}
\]
Therefore \( f(t) = C \) is a constant
\[
\text{At } t = t_0, \quad f(t_0) = 0 \implies C = 0
\]
\[
\implies f(t) = 0 \quad \forall \ t \in I
\]
which means \( \mathbf{x}(t) \) is orthogonal to \( \mathbf{V} \) for all \( t \in I \).

1.3 \( |\mathbf{x}(t)| \) is constant \( \iff |\mathbf{x}(t)|^2 \) is constant
\[
\iff \mathbf{x}(t) \cdot \mathbf{x}(t) \text{ is constant}
\]
\[
\implies (\mathbf{x}(t) \cdot \mathbf{x}(t))' = 0 \quad \forall \ t \in I
\]
\[
\implies 2 \mathbf{x}(t) \cdot \mathbf{x}'(t) = 0 \quad \forall \ t \in I
\]
\[
\implies \mathbf{x}(t) \text{ is orthogonal to } \mathbf{x}'(t) \quad \forall \ t \in I
\]

1.5 For simplicity let's study the point \( A \), which was at origin before the disk rolls.

Since the rolling has no slipping, at any time...

The displacement of \( O = \) the rotation of the disk times radius

Use the rotation angle \( \theta \) as our parameter. The coordinate of \( O \) is \((0, 1)\).
The vector $\overrightarrow{OA}$ is \((-\cos (\theta - \pi/2), \sin (\theta - \pi/2)) = (-\sin \theta, -\cos \theta)\).

Therefore, the coordinate of $A$ is \((\theta - \sin \theta, 1 - \cos \theta)\).

This is our desired curve.

(b) $x(\theta) = (\theta - \sin \theta, 1 - \cos \theta)$

$x'(\theta) = (1 - \cos \theta, \sin \theta)$

3) arc length of a complete rotation $\int_0^{2\pi} |x'(\theta)| d\theta = \int_0^{2\pi} \sqrt{2(1 - \cos \theta)} d\theta$

$= \int_0^{2\pi} \frac{2\sin \theta}{2} d\theta$

$= 4 \int_0^{\pi} \sin t \, dt = 8$.

1.6 (a) For any vector $V$ with $|V| = 1$

$(\theta - p) \cdot V = V \cdot \int_a^b x'(t) \, dt$

$= \int_a^b x'(t) \cdot V \, dt$ \quad \text{linearity of integral}

$\leq \int_a^b |x'(t)| \cdot |V| \cos \theta(t) \, dt$ \quad $\theta \text{ is the angle between } x'(t) \text{ and } V$

$\leq \int_a^b |x'(t)| \cdot (V) \, dt$ \quad $\cos \theta \leq 1$

$= \int_a^b |x'(t)| \, dt$ \quad $|V| = 1$
(b) Set \( V = \frac{\theta}{|\theta - \rho|} \) (assuming \( \rho \neq \theta \))

\[
(\theta - \rho) \cdot V = \frac{(\theta - \rho) \cdot (\theta - \rho)}{|\theta - \rho|} = \frac{|\theta - \rho|^2}{|\theta - \rho|} = |\theta - \rho| \\
\leq \int_a^b |a'(t)| \, dt \quad \text{as shown in (a)}
\]

If \( \theta = \rho \) then \( |\theta - \rho| = 0 \), \( \int_a^b |a'(t)| \, dt > 0 \)

In either case we have

\[
|\theta - \rho| \leq \int_a^b |a'(t)| \, dt
\]

the length of the length of some straight line arbitrary curve.

Therefore, the shortest path connecting any two pts in the straight line.

1.7

Let's study \( s \) and \( s + \Delta s \) two pts. Both \( \alpha'(s) \) and \( \alpha'(s + \Delta s) \) are unit vectors. Assuming the angle between them is \( \Delta \theta \) we have:

\[
|\alpha'(s + \Delta s) - \alpha'(s)| = 2 \sin \frac{\Delta \theta}{2}
\]

Hence

\[
|\alpha''(s)| = \lim_{\Delta s \to 0} \left| \frac{\alpha'(s + \Delta s) - \alpha'(s)}{\Delta s} \right| = \lim_{\Delta s \to 0} \frac{2 \sin \frac{\Delta \theta}{2}}{\Delta s}
\]

\[
= \frac{d\theta}{ds} \quad \text{the rate of change of the angle between tangents.}
\]
For a circle with radius $r$ we can write
$$\tilde{x}(t) = (r \cos t, r \sin t)$$
$$\tilde{x}'(t) = (-r \sin t, r \cos t)$$
$$\Rightarrow |\tilde{x}'(t)| = r$$
which means $\frac{ds}{dt} = r$.

Let $s = rt$, $t = \frac{s}{r}$ we have the arclength parameterization of the circle as
$$\alpha(s) = \left( r \cos \left( \frac{s}{r} \right), r \sin \left( \frac{s}{r} \right) \right)$$
$$\alpha''(s) = \left( -\frac{1}{r} \cos \left( \frac{s}{r} \right), -\frac{1}{r} \sin \left( \frac{s}{r} \right) \right)$$
$$\Rightarrow |\alpha''(s)| = \frac{1}{r}.$$