13. For a smooth curve $\alpha: I \rightarrow \mathbb{R}^2$. If at $t$, $\alpha'(t) \neq 0$ then we have a well defined unit tangent vector

$$T = \frac{\alpha'(t)}{|\alpha'(t)|}$$

Since it's in 2-d, given $T$, we have only two choices for $N$, corresponding to a orientation $s$ of $T-N$ frame. Let's pick $N$ to be $T$ rotating $\pi/2$ counter-clockwise.

To be explicit, if $T = (a, b)$ then $N = (-b, a)$

We can then define $k(s)$ as in

$$T'(s) = k(s) N(s).$$

Let the angle of $T$ w.r.t. x-axis be $\theta$. We have $k(s) = \theta'(s)$.

Of course we could have chosen $N$ to be the opposite direction. In that case $k$ measures the angular speed of $T$ vector in the clockwise direction. In this homework we'll stick to the counter-clockwise one.

14. As we mentioned in 13, $k(s) = \theta'(s)$.

Therefore $\theta(s) = \int_0^s k(s) \, ds + C_0$ for some $C_0$.

By definition of $\phi(s)$, $\alpha'(s) = \phi(s) = (C(s), \Theta(s), \sin \Theta(s))$

Thus $\alpha(s) = \left( \int_0^s C(s) \, ds + a, \int_0^s \sin \Theta(s) \, ds + b \right)$

for some constant $a, b$.

The fundamental theorem of calculus shows that the solution is unique up to the initial values $a, b, C_0$.

They correspond to the rigid transformations of the curve in $\mathbb{R}^2$. 
16. \( r = r(\theta) \Rightarrow x(\theta) = \begin{pmatrix} r(\theta) \cos \theta, \\ r(\theta) \sin \theta \end{pmatrix} \)

\[ x'(\theta) = \begin{pmatrix} r'(\theta) \cos \theta - r(\theta) \sin \theta, \\ r'(\theta) \sin \theta + r(\theta) \cos \theta \end{pmatrix} \]

\[ = r'(\theta) \begin{pmatrix} \cos \theta, \\ \sin \theta \end{pmatrix} + r(\theta) \begin{pmatrix} -\sin \theta, \\ \cos \theta \end{pmatrix} \]

Since \( \begin{pmatrix} \cos \theta, \\ \sin \theta \end{pmatrix} \perp \begin{pmatrix} -\sin \theta, \\ \cos \theta \end{pmatrix} \)

\[ |x'(\theta)|^2 = (r'^2 + r^2) \]

\[ \Rightarrow S = \int_a^b \sqrt{r'^2 + r^2} \, d\theta \]

(b) We use the definition that \( T'(s) = K(s) N(s) \)

To simplify notation, let \( \hat{\imath} = \begin{pmatrix} \cos \theta, \\ \sin \theta \end{pmatrix}, \hat{\jmath} = \begin{pmatrix} -\sin \theta, \\ \cos \theta \end{pmatrix} \)

\( \hat{\imath} \wedge \hat{\jmath} \) form an orthonormal frame. In this frame,

\[ x'(\theta) = r'(\theta) \hat{\imath} + r(\theta) \hat{\jmath} = |x'(\theta)| \cdot \hat{T} \]

\[ \Rightarrow N = \left( x'(\theta) \right)^{-1} \begin{pmatrix} \hat{\imath}', \\ \hat{\jmath}' \end{pmatrix} \]

On the other hand, \( \hat{\imath}' = \hat{\jmath}, \hat{\jmath}' = -\hat{\imath} \)

\[ \Rightarrow x''(\theta) = x'' \hat{\imath} + 2 x' \hat{\jmath} - x \hat{\imath} = (x'' - r) \hat{\imath} + 2 x' \hat{\jmath} \]

Therefore

\[ T'(s) = |x'(\theta)|^{-1} T'(\theta) = |x'(\theta)|^{-1} \left[ \frac{x''(\theta)}{|x'(\theta)|} - \frac{x'(\theta) \cdot x''(\theta)}{|x'(\theta)|^3} \right] \]

\[ K(s) = \langle T'(s), N(s) \rangle = |x'(\theta)|^{-3} \langle x''(\theta), \hat{\imath}' \rangle \]

\[ = |x'(\theta)|^{-3} \left( \hat{\imath}' \cdot (-x') \right) \]

\[ = \left( r^2 - r r'' + 2 r'^2 \right) \sqrt{r'^2 + r^2} \]

17. (a) As in 16 (a).

\[ T = \frac{x'(t)}{|x'(t)|} \quad T'(s) = |x'(t)|^{-1} \left[ \frac{x''(t)}{|x'(t)|} - \frac{x'(t) \cdot x''(t)}{|x'(t)|^3} \right] \]

\[ = K(s) N \]
Since \( T'(s) \perp T \) \& \( |T| = 1 \)

\[
K(s) = \left| T'(s) \right| \left| T(s) \times T'(s) \right|
\]

\[
= \left| \alpha'(t) \right|^{-3} \left| \alpha'(t) \times \alpha''(t) \right|
\]

Since \( \alpha'(t) \times \alpha'(t) = 0 \)

(b) We use the formula that

\[
T = \langle N'(s), B \rangle = |\alpha'(t)|^{-1} \langle N'(t), B \rangle
\]

Since \( B = T \times N = K(s)^{-1} \left( T \times T'(s) \right) = \frac{\alpha'(t) \times \alpha''(t)}{|\alpha'(t) \times \alpha''(t)|} \)

\[
\langle \alpha'(t), B(s) \rangle = \langle \alpha''(t), B(s) \rangle = 0.
\]

On the other hand

\[
N(t) = \frac{\alpha''(t) |\alpha'(t)|^2 - \alpha'(t) \langle \alpha'(t), \alpha''(t) \rangle}{|\alpha''(t) |\alpha'(t)|^2 - \alpha'(t) \langle \alpha'(t), \alpha''(t) \rangle}
\]

The only term in \( N'(t) \) that's not linear in \( \alpha'(t) \) & \( \alpha''(t) \) is

\[
\frac{\alpha'''(t) |\alpha'(t)|^2}{|\alpha''(t) |\alpha'(t)|^2 - \alpha'(t) \langle \alpha'(t), \alpha''(t) \rangle} = \frac{\alpha'''(t)}{|\alpha'(t)|^2} K
\]

\[
\Rightarrow \ T = |\alpha'(t)|^{-1} \langle N'(t), B \rangle = \frac{\langle \alpha'''(t), \alpha'(t) \times \alpha''(t) \rangle}{|\alpha'(t) \times \alpha''(t)|^2}
\]

Or in dot notation

\[
\frac{\alpha'''(t) \cdot (\alpha'(t) \times \alpha''(t))}{|\alpha'(t) \times \alpha''(t)|^2}.
\]