1. CURVES

Definition. A map

\[ F(x_1, \ldots, x_m) = (f_1(x_1,\ldots,x_m), \ldots, f_n(x_1,\ldots,x_m)) \]

from an open set in one Euclidean space into another Euclidean space is said to be smooth (or of class \( C^\infty \)) if it has continuous partial derivatives of all orders.
In this chapter, we will be dealing with *smooth curves*

\[ \alpha: I \to \mathbb{R}^3, \]

where \( I = (a, b) \) is an open interval in the real line \( \mathbb{R}^3 \), allowing \( a = -\infty \) or \( b = +\infty \).

Do Carmo calls these "*parametrized differentiable curves*", to emphasize that the specific function \( \alpha \) is part of the definition. Thus

\[ \alpha(t) = (\cos t, \sin t) \quad \text{and} \quad \beta(t) = (\cos 2t, \sin 2t) \]

are considered to be different curves in the plane, even though their *images* are the same circle.
Examples.

(1) The helix $\alpha(t) = (\cos t, \sin t, bt)$, $t \in \mathbb{R}$

(2) $\alpha(t) = (t^3, t^2)$.

Problem 1. Let $\alpha(t)$ be a smooth curve which does not pass through the origin. If $\alpha(t_0)$ is the point of its image which is closest to the origin (assuming such a point exists), and if $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to the velocity vector $\alpha'(t_0)$. 
**Problem 2.** Let \( \alpha: I \rightarrow \mathbb{R}^3 \) be a smooth curve and let \( V \in \mathbb{R}^3 \) be a fixed vector. Assume that \( \alpha'(t) \) is orthogonal to \( V \) for all \( t \in I \) and also that \( \alpha(t_0) \) is orthogonal to \( V \) for some \( t_0 \in I \). Prove that \( \alpha(t) \) is orthogonal to \( V \) for all \( t \in I \).

**Problem 3.** Let \( \alpha: I \rightarrow \mathbb{R}^3 \) be a smooth curve. Show that \( |\alpha(t)| \) is constant if and only if \( \alpha(t) \) is orthogonal to \( \alpha'(t) \) for all \( t \in I \).
Definition. A smooth curve $\alpha: I \rightarrow \mathbb{R}^3$ is said to be regular if $\alpha'(t) \neq 0$ for all $t \in I$. Equivalently, we say that $\alpha$ is an immersion of $I$ into $\mathbb{R}^3$.

The curve $\alpha(t) = (t^3, t^2)$ in the plane fails to be regular when $t = 0$.

A regular smooth curve has a well-defined tangent line at each point, and the map $\alpha$ is one-to-one on a small neighborhood of each point $t \in I$.

Convention. For simplicity, we'll begin omitting the word "smooth". So for example, we'll just say "regular curve", but mean "regular smooth curve".
Problem 4. If $\alpha: [a, b] \to \mathbb{R}^3$ is just continuous, and we attempt to define the arc length of the image $\alpha[a, b]$ to be the LUB of the lengths of all inscribed polygonal paths, show that this LUB may be infinite.

By contrast, show that if $\alpha$ is of class $C^1$ (meaning that it has a first derivative $\alpha'(t)$ which is continuous), then this LUB is finite and equals $\int_a^b |\alpha'(t)| \, dt$. 
Let $\alpha: I \to \mathbb{R}^3$ be a regular (smooth) curve. Then the arc length along $\alpha$, starting from some point $\alpha(t_0)$, is given by

$$s(t) = \int_{t_0}^{t} |\alpha'(t)| \, dt.$$ 

Note that $s'(t) = |\alpha'(t)| \neq 0$, so we can invert this function to obtain $t = t(s)$.

Then $\beta(s) = \alpha(t(s))$ is a reparametrization of our curve, and $|\beta'(s)| = 1$.

We will say that $\beta$ is \textit{parametrized by arc length}.
In what follows, we will generally parametrize our regular curves by arc length.

If \( \alpha: I \rightarrow \mathbb{R}^3 \) is parametrized by arc length, then the unit vector \( T(s) = \alpha'(s) \) is called the \textit{unit tangent vector} to the curve.
Problem 5. A circular disk of radius 1 in the xy-plane rolls without slipping along the x-axis. The figure described by a point of the circumference of the disk is called a cycloid.

(a) Find a parametrized curve $\alpha: \mathbb{R} \rightarrow \mathbb{R}^2$ whose image is the cycloid.

(b) Find the arc length of the cycloid corresponding to a complete rotation of the disk.
Problem 6. Let \( \alpha: [a, b] \to \mathbb{R}^3 \) be a parametrized curve, and set \( \alpha(a) = p \) and \( \alpha(b) = q \).

(1) Show that for any constant vector \( V \) with \( |V| = 1 \),

\[
(q - p) \cdot V = \int_a^b \alpha'(t) \cdot V \, dt \leq \int_a^b |\alpha'(t)| \, dt.
\]

(2) Set \( V = (q - p) / |q - p| \) and conclude that

\[
|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| \, dt.
\]

This shows that the curve of shortest length from \( \alpha(a) \) to \( \alpha(b) \) is the straight line segment joining these points.
Problem 7. Let $\alpha$: $I \rightarrow \mathbb{R}^3$ be parametrized by arc length. Thus the tangent vector $\alpha'(s)$ has unit length. Show that the norm $|\alpha''(s)|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at $s$.

Definition. If $\alpha$: $I \rightarrow \mathbb{R}^3$ is parametrized by arc length, then the number $\kappa(s) = |\alpha''(s)|$ is called the curvature of $\alpha$ at $s$. 
Problem 8. Show that the curvature of a circle is the reciprocal of its radius.

Let $\alpha: I \to \mathbb{R}^3$ be parametrized by arc length. When the curvature $\kappa(s) \neq 0$, the unit vector

$$N(s) = \frac{\alpha''(s)}{|\alpha''(s)|}$$

is well-defined.

Problem 9. Show that the unit vector $N(s)$ is normal to the curve, in the sense that $N(s) \cdot T(s) = 0$, where $T(s)$ is the unit tangent vector to the curve.

Definition. When $\kappa(s) \neq 0$, we call $N(s)$ the principal normal vector to the curve.
Let $\alpha: I \to \mathbb{R}^3$ be parametrized by arc length, and let $T(s)$ be the unit tangent vector along $\alpha$.

If the curvature $\kappa(s) \neq 0$, then we also have the principal normal vector $N(s)$ at $\alpha(s)$.

In that case, define the **binormal vector** $B(s)$ to $\alpha$ at $s$ by the vector cross product,

$$B(s) = T(s) \times N(s).$$

**Problem 10.** Show that $B'(s)$ is parallel to $N(s)$. 
Definition. If $\kappa(s) \neq 0$, the \textit{torsion} $\tau(s)$ of the curve $\alpha$ at $s$ is defined by the formula

$$B'(s) = -\tau(s) N(s).$$

This is the opposite sign convention from do Carmo.

Problem 11. Find the curvature and torsion of the helix $\alpha(t) = (a \cos t, a \sin t, b t)$.
Problem 12. Let $\alpha: I \to \mathbb{R}^3$ be parametrized by arc-length and have nowhere vanishing curvature $\kappa(s) \neq 0$. Show that

\[ T'(s) = \kappa(s) N(s) \]

\[ N'(s) = -\kappa(s) T(s) + \tau(s) B(s) \]

\[ B'(s) = -\tau(s) N(s). \]
Definition. The above equations are called the \textit{Frenet equations}, and the orthonormal frame

\[ T(s), \ N(s), \ B(s) \]

is called the \textbf{Frenet frame} along the curve $\alpha$. 
THEOREM. Given smooth functions $\kappa(s) > 0$ and $\tau(s)$, for $s \in I$, there exists a regular curve $\alpha: I \rightarrow \mathbb{R}^3$ parametrized by arc length, with curvature $\kappa(s)$ and torsion $\tau(s)$.

Moreover, another other such curve $\beta: I \rightarrow \mathbb{R}^3$ differs from $\alpha$ by a rigid motion of $\mathbb{R}^3$.

This result is sometimes called the

fundamental theorem of the local theory of curves.
Problem 13. The curvature of a smooth curve in the plane can be given a well-defined sign, just like the torsion of a curve in 3-space. Explain why this is so.

Problem 14. Given a smooth function $\kappa(s)$ defined for $s$ in the interval $I$, show that the arc-length parametrized plane curve having $\kappa(s)$ as curvature is given by

$$\alpha(s) = \left( \int \cos \theta(s) \, ds + a, \int \sin \theta(s) \, ds + b \right),$$

where

$$\theta(s) = \int \kappa(s) \, ds + \theta_0.$$  

Show that this solution is unique up to translation by $(a, b)$ and rotation by $\theta_0$. 
Proof of the fundamental theorem of the local theory of curves in \( \mathbb{R}^3 \).

We are given smooth functions \( \kappa(s) > 0 \) and \( \tau(s) \), for \( s \in I \), and must find a regular curve \( \alpha : I \to \mathbb{R}^3 \) parametrized by arc length, with curvature \( \kappa(s) \) and torsion \( \tau(s) \).

Let's begin by writing the Frenet equations for the Frenet frame.

\[
\begin{align*}
T'(s) &= \kappa(s) N(s) \\
N'(s) &= -\kappa(s) T(s) + \tau(s) B(s) \\
B'(s) &= -\tau(s) N(s).
\end{align*}
\]
We'll view this as a system of three first order linear ODEs, with given coefficients $\kappa(s)$ and $\tau(s)$, for the unknown Frenet frame $T(s)$, $N(s)$, $B(s)$.

We can also view it as a system of nine first order linear ODEs for the components of the Frenet frame.
Now the fundamental existence and uniqueness theorem for systems of first order ODEs promises a unique "local solution", that is, a solution defined in some unspecified neighborhood of any given point \( s_0 \in I \), with preassigned "initial conditions" \( T(s_0) \), \( N(s_0) \), \( B(s_0) \).

Although for general systems we can only guarantee a local solution, for linear systems another theorem promises a unique "global solution", that is, one defined on the entire interval \( I \).
So we'll use that theorem, pick an arbitrary point $s_0 \in I$, and pick an arbitrary "right handed" orthonormal frame $T(s_0), N(s_0), B(s_0)$ to get us started.

Then we'll apply the global existence and uniqueness theorem for linear systems of ODEs to get a unique family of vectors $T(s), N(s), B(s)$ which are defined for all $s \in I$, which satisfy the Frenet equations, and which have arbitrary preassigned initial values $T(s_0), N(s_0), B(s_0)$. 
Let's pause to check that the nature of the Frenet equations

\[ T'(s) = \kappa(s) N(s) \]

\[ N'(s) = -\kappa(s) T(s) + \tau(s) B(s) \]

\[ B'(s) = -\tau(s) N(s) \]

guarantees that if we start off with an orthonormal frame \( T(s_0) , N(s_0) , B(s_0) \), then the solution will be an orthonormal frame for all \( s \in I \).
Consider the six real-valued functions defined for $s \in I$, and obtained by taking the various inner products of the vectors $T(s)$, $N(s)$, $B(s)$:

\[
\begin{align*}
&T(s), T(s) > \\
&N(s), N(s) > \\
&B(s), B(s) > \\
&T(s), N(s) > \\
&T(s), B(s) > \\
&N(s), B(s) > .
\end{align*}
\]

When $s = s_0$, these six quantities start off with the values

\[
\begin{array}{ccc}
1 & 1 & 1 \\
0 & 0 & 0 \\
\end{array}
\]
These six quantities satisfy a system of first order linear ODEs, obtained from the Frenet equations. For example,

\[
\langle T(s), T(s) \rangle' = 2 \langle T(s), T'(s) \rangle = 2 \kappa(s) \langle T(s), N(s) \rangle
\]

\[
\langle T(s), N(s) \rangle' = \langle T'(s), N(s) \rangle + \langle T(s), N'(s) \rangle = \langle \kappa(s) N(s), N(s) \rangle + \langle T(s), -\kappa(s) T(s) + \tau(s) B(s) \rangle
\]

\[
= \kappa(s) \langle N(s), N(s) \rangle - \kappa(s) \langle T(s), T(s) \rangle + \tau(s) \langle T(s), B(s) \rangle,
\]
and so forth.
The constant solution

\[ <T(s), T(s)> = 1 \quad <N(s), N(s)> = 1 \quad <B(s), B(s)> = 1 \]

\[ <T(s), N(s)> = 0 \quad <T(s), B(s)> = 0 \quad <N(s), B(s)> = 0 \]

satisfies this system of ODEs, with the given initial conditions, so by uniqueness this is the only solution.

**Conclusion:** If the vectors \( T(s), N(s), B(s) \) start out orthonormal at \( s_0 \in I \), then they remain orthonormal for all \( s \in I \).
Where are we so far?

We have proved that, given smooth functions $\kappa(s) > 0$ and $\tau(s)$ defined for all $s \in I$, and an orthonormal frame $T(s_0), N(s_0), B(s_0)$ defined for some $s_0 \in I$, then there is a unique orthonormal frame $T(s), N(s), B(s)$ defined for all $s \in I$ with these preassigned initial values, and satisfying the Frenet equations throughout $I$. 
Now, to get the curve $\alpha: I \to \mathbb{R}^3$ defined for $s \in I$ and having the preassigned curvature $\kappa(s) > 0$ and torsion $\tau(s)$, just pick the point $\alpha(s_0)$ at random in $\mathbb{R}^3$ and then define

$$\alpha(s) = \alpha(s_0) + \int_{s_0}^{s} T(t) \, dt.$$ 

We get $\alpha'(s) = T(s)$, which is a unit vector, so $\alpha$ is parametrized by arc-length.
The Frenet equations

\[ T'(s) = \kappa(s) N(s) \]

\[ N'(s) = -\kappa(s) T(s) + \tau(s) B(s) \]

\[ B'(s) = -\tau(s) N(s) \]

then tell us that the curve \( \alpha \) has curvature \( \kappa(s) \) and torsion \( \tau(s) \), as desired.
Once the point $\alpha(s_0)$ and the initial orthonormal frame $T(s_0), N(s_0), B(s_0)$ is picked, the curve is unique.

Thus any other such curve $\beta: I \rightarrow \mathbb{R}^3$ differs from $\alpha$ by a rigid motion of $\mathbb{R}^3$.

This completes the proof of the fundamental theorem of the local theory of curves in $\mathbb{R}^3$. 
Problem 15. Let $\alpha: I \to \mathbb{R}^3$ be a regular curve with nowhere vanishing curvature. Assume that all the principal normal lines of $\alpha$ pass through a fixed point in $\mathbb{R}^3$. Prove that the image of $\alpha$ lies on a circle.

Problem 16. Let $r = r(\theta)$, $a \leq \theta \leq b$, describe a plane curve in polar coordinates. 

(a) Show that the arc length of this curve is given by

$$\int_a^b [r^2 + (r')^2]^{1/2} \, d\theta .$$

(b) Show that the curvature is given by

$$\kappa(\theta) = \frac{[2(r')^2 - r r'' + r^2]}{[(r')^2 + r^2]^{3/2}} .$$
Problem 17. Let \( \alpha: I \to \mathbb{R}^3 \) be a regular curve, not necessarily parametrized by arc length.

(a) Show that the curvature of \( \alpha \) is given by

\[
\kappa(t) = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.
\]

(b) If the curvature is nonzero, so that the torsion is well-defined, show that the torsion is given by

\[
\tau(t) = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2}.
\]
Definitions. A *closed plane curve* is a regular curve \( \alpha: [a, b] \to \mathbb{R}^2 \) such that \( \alpha \) and all its derivatives agree at \( a \) and at \( b \), that is,

\[
\alpha(a) = \alpha(b), \quad \alpha'(a) = \alpha'(b), \quad \alpha''(a) = \alpha''(b), \ldots
\]

Alternatively, one can use the entire real line as domain, \( \alpha: \mathbb{R} \to \mathbb{R}^2 \), and require that \( \alpha \) be periodic of some period \( L > 0 \), that is, \( \alpha(t + L) = \alpha(t) \) for all \( t \in \mathbb{R} \).

Another alternative: one can use a circle (of any radius) as the domain for a closed curve.
A curve is *simple* if it has no further intersections, other than the coincidence of the beginning and end points.

If we use a circle for the domain, \( \alpha: S^1 \to \mathbb{R}^2 \), then the curve is simple if \( \alpha \) is one-to-one. Since \( S^1 \) is compact, this is the same thing as saying that \( \alpha \) is a homeomorphism onto its image.
If $\alpha: [a, b] \to \mathbb{R}^2$ is a regular closed curve in the plane, parametrized by arc length, then its *total curvature* is defined by the integral

$$\text{Total curvature} = \int_a^b \kappa(s) \, ds.$$
Problem 18. (a) Show that the total curvature of a regular closed curve in the plane is \(2n\pi\) for some integer \(n\).

(b) Show that if the regular closed curve is simple, then \(n = +1\) or \(-1\).

(c) Suppose that a regular closed curve in the plane has curvature which is strictly positive or strictly negative, and that the above integer \(n\) equals \(+1\) or \(-1\).

Show that the curve is simple.
Let $\alpha: S^1 \to \mathbb{R}^2$ be a regular closed curve in the plane. For each point $\theta \in S^1$, the unit tangent vector $T(\theta)$ to the curve at the point $\alpha(\theta)$ is given by

$$T(\theta) = \frac{\alpha'(\theta)}{|\alpha'(\theta)|}.$$ 

Thus $T: S^1 \to S^1$, and then the induced map

$$T_*: \pi_1(S^1) \to \pi_1(S^1)$$

is a group homomorphism from the integers to the integers, and hence is multiplication by some integer $n$, which we call the \textit{degree} of the map $T$, or the \textit{winding number} or \textit{rotation index} of the curve $\alpha$. 
Problem 19. Show that this integer $n$ is the same as the integer $n$ in the previous problem, that is, show that the total curvature of the curve $\alpha$ is $2\pi n$.

Definition. Let $\alpha_0$ and $\alpha_1 : S^1 \to \mathbb{R}^2$ be regular closed curves in the plane. A homotopy

$$A : S^1 \times [0, 1] \to \mathbb{R}^2$$

between $\alpha_0$ and $\alpha_1$ is said to be a **regular homotopy** if each intermediate curve, $\alpha_t : S^1 \to \mathbb{R}^2$, defined by $\alpha_t(\theta) = A(\theta, t)$, is a regular curve.
**Remark.** If $\alpha_0$ and $\alpha_1 : S^1 \rightarrow \mathbb{R}^2$ are regularly homotopic, then they have the same winding number.

**WHITNEY-GRAUSTEIN THEOREM.** Two regular curves $\alpha_0$ and $\alpha_1 : S^1 \rightarrow \mathbb{R}^2$ are regularly homotopic if and only if they have the same winding number.
Volumes of tubes...two problems.

(1) Show that the area of a tube of radius \( \varepsilon \) about a simple closed curve of length \( L \) in the plane is \( 2\varepsilon L \).

(2) Show that the volume of a tube of radius \( \varepsilon \) about a simple closed curve of length \( L \) in 3-space is \( \pi \varepsilon^2 L \).

We will solve both of these problems, and the Frenet equations for curves will be our main tool.
Tubes about circles in the plane.

The simplest example is that of a tube of radius $\varepsilon$ about a circle of radius $r$ in the plane, so just an annulus between concentric circles of radii $r + \varepsilon$ and $r - \varepsilon$, with area

$$\pi (r + \varepsilon)^2 - \pi (r - \varepsilon)^2 = \pi 4r\varepsilon = (2\pi r)(2\varepsilon)$$

$$= (\text{circumference of circle}) \cdot (\text{width of tube})$$
Tubes about any curve in the plane.

Parametrize the curve by arc length: \( x = x(s) \) for \( 0 \leq s \leq L \).

Let \( T(s) = x'(s) \) and \( N(s) \) denote unit tangent and normal vectors along the curve.

Frenet eqns: \( T'(s) = \kappa(s) N(s) \) and \( N'(s) = -\kappa(s) T(s) \).
To produce the $\varepsilon$-tube about this curve, we define

$$F: \{0 \leq s \leq L\} \times \{-\varepsilon < t < \varepsilon\} \to \mathbb{R}^2$$

by

$$F(s, t) = x(s) + t N(s) .$$

Then the partial derivatives of $F$ are given by

$$F_s = x'(s) + t N'(s) = T(s) + t (-\kappa(s) T(s))$$

$$= (1 - t \kappa(s)) T(s)$$

$$F_t = N(s) .$$
Hence the area of the $\varepsilon$-tube about our curve is given by

$$
\int_s \int_t |\det dF| \ dt \ ds = \int_s \int_t (1 - t \kappa(s)) \ dt \ ds
$$

$$
= \int_s (t - 1/2 \ t^2 \kappa(s)) \|_{-\varepsilon}^{\varepsilon} \ ds = \int_s 2\varepsilon \ ds = L \cdot 2\varepsilon
$$

$$
= \text{(length of curve) \ (width of strip)},
$$

independent of the nature of the curve.
Tubes about any curve in 3-space.

Parametrize the curve by arc length: \( x = x(s) \) for \( 0 \leq s \leq L \).

Frenet frame along the curve: \( T(s) = x'(s) \), \( N(s) \), \( B(s) \).

Frenet eqns:
\[
T'(s) = \kappa(s) N(s)
\]
\[
N'(s) = -\kappa(s) T(s) + \tau(s) B(s)
\]
\[
B'(s) = -\tau(s) N(s)
\]
To produce the $\varepsilon$-tube about this curve, we define

$$F: \{0 \leq s \leq L\} \times \{t^2 + u^2 < \varepsilon\} \rightarrow \mathbb{R}^3$$

by

$$F(s, t, u) = x(s) + t N(s) + u B(s).$$

Then the partial derivatives of $F$ are given by

$$F_s = (1 - t \kappa(s) T(s) - u \tau(s) N(s) + t \tau(s) B(s))$$

$$F_t = N(s)$$

$$F_u = B(s)$$
Hence the volume of the $\varepsilon$-tube about our curve is given by

$$
\int_s \int_{t^2+u^2<\varepsilon^2} |\text{det } dF| \, dt \, du \, ds = \int_s \int_{t^2+u^2<\varepsilon^2} (1 - t \kappa(s)) \, dt \, du \, ds
$$

$$
= \int_s \pi \varepsilon^2 \, ds = L \cdot \pi \varepsilon^2
$$

$$
= (\text{length of curve}) \, (\text{area of } \varepsilon\text{-disk}),
$$

independent of the nature of the curve.

We used the fact that the integral of the odd function $t$ over the disk $t^2 + u^2 < \varepsilon^2$ is zero.
**Problem.** To get a Frenet frame along a curve in $\mathbb{R}^3$, one needs to assume that the curvature $\kappa(s)$ never vanishes.

Without this hypothesis, one can still prove that

$$\text{vol } \varepsilon\text{-tube} = (\text{length of curve}) (\text{area of } \varepsilon\text{-disk}).$$

(a) Let $T(s), A(s), B(s)$ be an ON frame along our curve $x(s)$. Show that the Frenet eqns are replaced by

$$T'(s) = \alpha(s) A(s) + \beta(s) B(s)$$

$$A'(s) = -\alpha(s) T(s) + \gamma(s) B(s)$$

$$B'(s) = -\beta(s) T(s) - \gamma(s) A(s).$$
(b) Defining the $\varepsilon$-tube about our curve $x(s)$ by

$$F(s, t, u) = x(s) + t A(s) + u B(s),$$

with $T(s) = x'(s)$, and hence $A(s)$ and $B(s)$ orthogonal to the curve, show that we get

$$\text{vol } \varepsilon\text{-tube} = \int_s \int_{t^2 + u^2 < \varepsilon^2} |\det dF| \, dt \, du \, ds$$

$$= \int_s \int_{t^2 + u^2 < \varepsilon^2} (1 - t \alpha(s) - u \beta(s)) \, dt \, du \, ds$$

$$= \int_s \pi \varepsilon^2 \, ds = L \cdot \pi \varepsilon^2$$

$$= (\text{length of curve}) \times (\text{area of } \varepsilon\text{-disk}).$$
**Tough Problem.** Show that for a smooth curve in $\mathbb{R}^n$, we get

$$\text{vol } \varepsilon\text{-tube} = (\text{length of curve}) \cdot (\text{vol } B^{n-1}(\varepsilon)),$$

where $B^{n-1}(\varepsilon)$ is a round ball of radius $\varepsilon$ in $\mathbb{R}^{n-1}$. 
Tubes about round spheres in 3-space.
The "tube" of radius $\varepsilon$ about a round sphere of radius $r$ in 3-space is just the region between the concentric spheres of radii $r + \varepsilon$ and $r - \varepsilon$, with volume

$$4/3 \pi (r + \varepsilon)^3 - 4/3 \pi (r - \varepsilon)^3 = 4/3 \pi (6 r^2 \varepsilon + 2 \varepsilon^3)$$

$$= (4 \pi r^2) 2\varepsilon + 8/3 \pi \varepsilon^3 = (\text{area of sphere}) 2\varepsilon + 8/3 \pi \varepsilon^3$$

$$= 2\varepsilon (\text{area of sphere} + 2\pi/3 \chi(\text{sphere}) \varepsilon^2) ,$$

which is exactly "Weyl's tube formula" for surfaces in $\mathbb{R}^3$. 
Problem. Compute the volume of the $\varepsilon$-tube about a torus of revolution in 3-space, and show that it is

$$\text{vol } \varepsilon\text{-tube} = 2\varepsilon \left(\text{area of torus}\right)$$

$$= 2\varepsilon \left(\text{area of torus} + 2\pi/3 \chi(\text{torus}) \varepsilon^2\right),$$

since $\chi(\text{torus}) = 0$, again in accord with Weyl's tube formula.