Homework 10 Solutions

1. (a) We will show that $\Phi$ is a homomorphism, injective, and surjective. To see that it is a homomorphism, let $a, b \in R$ and $\phi, \psi \in \text{hom}(M, N)$. We want to show that $\Phi(a\phi + b\psi) = a\Phi(\phi) + b\Phi(\psi)$. Let $f \in N^\ast$. Then

$$\Phi(a\phi + b\psi)(f) = (a\phi + b\psi)^\ast(f) = f \circ (a\phi + b\psi) = f \circ (a\phi) + f \circ (b\psi) = a(f \circ \phi) + b(f \circ \psi) = a\phi^\ast(f) + b\psi^\ast(f) = a\Phi(\phi)(f) + b\Phi(\psi)(f)$$

as desired.

For isomorphism, we look at part b to show that $\Phi$ is transpose, and then problem 2 to see that transpose is an isomorphism.

(b) Let $\phi : M \to N$ be a homomorphism, and let $A_\phi$ the matrix representing it. That is, the matrix with entries $a_{ij}$ so that $\phi(\alpha_i) = \sum_{j=1}^n a_{ij}\beta_j$. The matrix of $\phi^\ast$ has elements $b_{ki}$ given by $\phi^\ast(\beta_k^i) = \sum_{j=1}^n b_{kj}\alpha_{ji}$. The latter is $\beta_k^i \circ \phi$, and we apply this to $\alpha_i$ and obtain $\beta_k^i(\phi(\alpha_i)) = \beta_k^i(\sum_{j=1}^n a_{ij}\beta_j) = a_{ik}$. But similarly, it is $\phi^\ast(\beta_k^i)(\alpha_i) = \sum_{j=1}^n b_{kj}\alpha_{ji}(\alpha_i) = b_{ki}$. And so $a_{ik} = b_{ki}$, so $A_{\phi^\ast} = A_\phi^\ast$.

2. (a) We must show homomorphism, injective, and surjective. To see that it is a homomorphism, let $A, B$ be $n \times m$ matrices, and let $\alpha, \beta \in R$. We want to show that $(\alpha A + \beta B)^\tau = \alpha A^\tau + \beta B^\tau$. We do this be noting that the matrix elements of $\alpha A + \beta B$ are $\alpha a_{ij} + \beta b_{ij}$, and the transpose has $\alpha a_{ji} + \beta b_{ji}$. The Similarly, for $\alpha A^\tau + \beta B^\tau$, we have $\alpha a_{ji} + \beta b_{ji}$, and so the two are equal. For injectivity and surjectivity, we will show that transpose is its own inverse. Let $A$ be a matrix with elements $a_{ij}$. Then its transpose is a matrix with elements $a_{ji}$, and its transpose is $a_{ij}$ again. Thus, $A^{\tau \tau} = A$, and so transpose is its own inverse, and so must be an isomorphism.

(b) We have shown it to be a module isomorphism. All that remains is to show that $(AB)^\tau = B^\tau A^\tau$. Let $A$ and $B$ be matrices with elements $a_{ij}$ and $b_{ij}$. Then $AB$ has matrix elements $\sum_{k=1}^n a_{ik}b_{kj}$.
The transpose is $\sum_{k=1}^{m} a_{jk} b_{ki}$, so these are the matrix elements of $(AB)^T$. For $A^T$ and $B^T$, we have elements $a_{ji}$ and $b_{ij}$. Multiplying together we get $\sum_{k=1}^{m} a_{jk} b_{ki}$, as desired.

3. (a) If $A$ is a matrix with elements $a_{ij}$, $\text{tr}(A) = \sum_{i=1}^{m} a_{ii}$. So, if $A, B$ are arbitrary matrices with elements $a_{ij}$ and $b_{ij}$, and $r \in R$, then $A + B$ has elements $a_{ij} + b_{ij}$, and so $\text{tr}(A + B) = \sum_{i=1}^{m} a_{ii} + b_{ii} = \text{tr}(A) + \text{tr}(B)$, and $rA$ has elements $ra_{ij}$, so $\text{tr}(rA) = \sum_{i=1}^{m} ra_{ii} = r \sum_{i=1}^{m} a_{ii} = r \text{tr}(A)$.

(b) False. Let $I$ be the $2 \times 2$ identity matrix. Then $\text{tr}(I \cdot I) = \text{tr}(I) = 2$, but $\text{tr}(I) \cdot \text{tr}(I) = 2 \times 2 = 4$.

(c) Let $A$ be an arbitrary matrix with elements $a_{ij}$. Then $\text{tr}(A) = \sum_{i=1}^{m} a_{ii} = \sum_{j=1}^{m} a_{ij} = \text{tr}(A^T)$.

(d) Let $A, B$ be matrices with elements $a_{ij}, b_{ij}$. Then $AB$ has elements $\sum_{k=1}^{m} a_{ik} b_{kj}$, and $BA$ has elements $\sum_{k=1}^{m} b_{ik} a_{kj}$. Taking traces, we have $\text{tr}(AB) = \sum_{i=1}^{m} \sum_{k=1}^{m} a_{ik} b_{ki}$, and $\text{tr}(BA) = \sum_{i=1}^{m} \sum_{k=1}^{m} b_{ik} a_{ki}$. These are equal by relabeling the indices.

(e) Let $S \in GL_m(R)$ and $A$ be a matrix. Then $\text{tr}(SAS^{-1}) = \text{tr}(S(AS^{-1}))$. By the previous part, this is equal to $\text{tr}((AS^{-1})S) = \text{tr}(ASS^{-1}) = \text{tr}(A)$.

(f) False. The invertible matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has trace zero.

4. For this problem, we must merely check these things for $P_A(X)$, because $P_\phi(X)$ is defined to be $P_A(x)$ for some $A$ representing $\phi$.

(a) We proceed by induction. As $P_A(X) = \det(XI - A)$, for a $1 \times 1$ matrix $A = (a)$, this is $x - a$, a degree 1 polynomial. Now, assume that the result holds for matrices of size less than $k \times k$, and let $A$ be a $k \times k$ matrix. $A_{ij}$ the minor obtained by deleting the $i$th row and $j$th column, and $\Delta_{ij}$ the similar minor of $XI - A$. Then, $P_A(X) = (X - a_{11}) \det(XI - A_{11}) + \sum_{i=2}^{k} (-1)^{i+1} a_{i1} \Delta_{i1}$. Now, $\Delta_{i1}$ is of degree less than or equal to $k - 2$, because it has had two of the $x - a_{jj}$ terms removed. By induction hypothesis, $\det(XI - A_{11})$ is monic of degree $k - 1$, and so $(x - a_{11}) \det(XI - A_{11})$ is monic of degree $k$.

(b) Obtaining $a_0$ is simple. We have $P_A(0) = a_0$, and, as $P_A(X) = \det(XI - A)$, this means that $a_0 = P_A(0) = \det(0I - A) = \det(-A)$. To obtain $a_{m-1}$, we use induction. For a $1 \times 1$ matrix, it can be seen to be $-\text{tr}(A)$. We assume that this holds for matrices up to $m \times m$. Now, from before, we know that $P_A(X) = (X - a_{11}) \det(XI - A_{11}) + \sum_{i=2}^{m} (-1)^{i+1} a_{i1} \Delta_{i1}$. However, none of these tail terms contribute to $a_{m-1}$, as their degree is too low. It is the $a_{m-1}$ term of $(X - a_{11}) \det(XI - A_{11})$. By induction hypothesis, $\det(XI - A_{11})$ leads with $X^{m-1} = \text{tr}(A_{11})X^{m-2}$. Multiplying through, we get that $a_{m-1} = -a_{11} - \text{tr}(A_{11}) = -\text{tr}(A)$, as desired.
(c) Let $A$ be a matrix. Then $P_A(X) = \det(XI - A) = \det(XI - A)^\tau = \det(XI^\tau - A^\tau) = \det(XI - A^\tau) = P_A^\tau(X)$.

(d) Let $S$ invertible and $A$ arbitrary. Then $P_{SAS^{-1}}(X) = \det(XI - SAS^{-1}) = \det(XSIS^{-1} - SAS^{-1}) = \det(SXI - A)S^{-1}) = \det(XI - A) = P_A(X)$.

5. (a) The characteristic polynomial is $x^4 - 1$, and so we get four eigenvalues, $1, -1, i, -i$ with corresponding eigenvectors

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
1 \\
-1 \\
-1 \\
1
\end{pmatrix}, \begin{pmatrix}
i \\
-1 \\
-i \\
1
\end{pmatrix}, \begin{pmatrix}
i \\
1 \\
1 \\
1
\end{pmatrix}
\]

when working over a field where $-1$ has square roots $\pm i$. If $-1$ is not square, then there are only two eigenvalues and eigenvectors.

(b) The characteristic polynomial is $x^4 - 16x^3 + 90x^2 - 216x + 189 = (x-3)^3(x-7)$. So it has four eigenvalues, 3, 3, 3, 7, with eigenvectors

\[
\begin{pmatrix}
-1 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
-1 \\
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
-1 \\
0 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\]

which works over any field at all.

(c) For this one, the formulas are very complicated, and can be obtained by careful calculation and application of the cubic formula.