Homework 10 Solutions

1. (a) We will show that Φ is a homomorphism, injective, and surjective. To see that it is a homomorphism, let $a, b \in R$ and $\phi, \psi \in \hom(M, N)$. We want to show that $\Phi(a\phi + b\psi) = a\Phi(\phi) + b\Phi(\psi)$. Let $f \in N^*$. Then

$$\begin{aligned}\Phi(a\phi + b\psi)(f) &= (a\phi + b\psi)^*(f) \\ &= f \circ (a\phi + b\psi) \\ &= f \circ (a\phi) + f \circ (b\psi) \\ &= a(f \circ \phi) + b(f \circ \psi) \\ &= a\phi^*(f) + b\psi^*(f) \\ &= a\Phi(\phi)(f) + b\Phi(\psi)(f)\end{aligned}$$

as desired.

For isomorphism, we look at part b to show that Φ is transpose, and and then problem 2 to see that transpose is an isomorphism.

(b) Let $\phi : M \to N$ be a homomorphism, and let A_ϕ the matrix representing it. That is, the matrix with entries a_{ij} so that $\phi(\alpha_i) = \sum_{j=1}^{n} a_{ij}\beta_j$. The matrix of ϕ^* has elements b_{kl} given by $\phi^*(\beta_k^*) = \sum_{l=1}^{m} b_{kl}\alpha_l^*$. The latter is $\beta_k^* \circ \phi$, and we apply this to α_i and obtain $\beta_k^*(\phi(\alpha_i)) = \beta_k^*(\sum_{j=1}^{n} a_{ij}\beta_j) = a_{ik}$. But similarly, it is $\phi^*(\beta_k^*)(\alpha_i) = \sum_{l=1}^{m} b_{kl}\alpha_l^*(\alpha_i) = b_k i$. And so $a_{ik} = b_{ki}$, so $A_{\phi^*} = A_\phi^\tau$.

2. (a) We must show homomorphism, injective, and surjective. To see that it is a homomorphism, let A, B be $n \times m$ matrices, and let $\alpha, \beta \in R$. We want to show that $(\alpha A + \beta B)^\tau = \alpha A^\tau + \beta B^\tau$. We do this be noting that the matrix elements of $aA + bB$ are $\alpha a_{ij} + \beta b_{ij}$, and the transpose has $\alpha a_{ji} + \beta b_{ji}$. The Similarly, for $\alpha A^\tau + \beta B^\tau$, we have $\alpha a_{ji} + \beta b_{ji}$, and so the two are equal. For injectivity and surjectivity, we will show that transpose is its own inverse. Let A be a matrix with elements a_{ij}. Then its transpose is a matrix with elemetns a_{ji}, and its transpose is a_{ij} again. Thus, $A^{\tau\tau} = A$, and so transpose is its own inverse, and so must be an isomorphism.

(b) We have shown it to be a module isomorphism. All that remains is to show that $(AB)^\tau = B^\tau A^\tau$. Let A and B be matrices with elements a_{ij} and b_{ij}. Then AB has matrix elements $\sum_{k=1}^{m} a_{ik}b_{kj}$.

The transpose is $\sum_{k=1}^{m} a_{jk}b_{ki}$, so these are the matrix elements of $(AB)^\tau$. For A^τ and B^τ, we have elements a_{ji} and b_{ji}. Multiplying together we get $\sum_{k=1}^{m} a_{jk}b_{ki}$, as desired.

3. (a) If A is a matrix with elements a_{ij}, $\operatorname{tr}(A) = \sum_{i=1}^{m} a_{ii}$. So, if A, B are arbitrary matrices with elements a_{ij} and b_{ij}, and $r \in R$, then $A + B$ has elements $a_{ij} + b_{ij}$, and so $\operatorname{tr}(A + B) = \sum_{i=1}^{m} a_{ii} + b_{ii} = \operatorname{tr}(A) + \operatorname{tr}(B)$, and rA has elements ra_{ij}, so $\operatorname{tr}(rA) = \sum_{i=1}^{m} ra_{ii} = r\sum_{i=1}^{m} a_{ii} = r\operatorname{tr}(A)$.

   (b) False. Let I be the 2×2 identity matrix. Then $\operatorname{tr}(I * I) = \operatorname{tr}(I) = 2$, but $\operatorname{tr}(I)\operatorname{tr}(I) = 2 * 2 = 4$.

   (c) Let A be an arbitrary matrix with elements a_{ij}. Then $\operatorname{tr}(A) = \sum_{i=1}^{m} a_{ii} = \sum_{j=1}^{m} a_{jj} = \operatorname{tr}(A^\tau)$.

   (d) Let A, B be matrices with elements a_{ij}, b_{ij}. Then AB has elements $\sum_{k=1}^{m} a_{ik}b_{kj}$, and BA has elements $\sum_{k=1}^{m} b_{ik}a_{kj}$. Taking traces, we have $\operatorname{tr}(AB) = \sum_{i=1}^{m}\sum_{k=1}^{m} a_{ik}b_{ki}$, and $\operatorname{tr}(BA) = \sum_{i=1}^{m}\sum_{k=1}^{m} b_{ik}a_{ki}$. These are equal by relabeling the indices.

   (e) Let $S \in GL_m(R)$ and A be a matrix. Then $\operatorname{tr}(SAS^{-1}) = \operatorname{tr}(S(AS^{-1}))$. By the previous part, this is equal to $\operatorname{tr}((AS^{-1})S) = \operatorname{tr}(ASS^{-1}) = \operatorname{tr}(A)$.

   (f) False. The invertible matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has trace zero.

4. For this problem, we must merely check these things for $P_A(X)$, because $P_\phi(X)$ is defined to be $P_A(x)$ for some A representing ϕ.

   (a) We proceed by induction. As $P_A(X) = \det(XI - A)$, for a 1×1 matrix $A = (a)$, this is $x - a$, a degree 1 polynomial. Now, assume that the result holds for matrices of size less than $k \times k$, and let A be a $k \times k$ matrix, A_{ij} the minor obtained by deleting the ith row and jth column, and Δ_{ij} the similar minor of $XI - A$. Then, $P_A(X) = (X - a_{11})\det(XI - A_{11}) + \sum_{i=2}^{k}(-1)^{i+1}a_{1i}\Delta_{1i}$. Now, Δ_{1i} is of degree less than or equal to $k-2$, because it has had two of the $x - a_{jj}$ terms removed. By induction hypothesis, $\det(XI - A_{11})$ is monic of degree $k-1$, and so $(x - a_{11})\det(XI - A_{11})$ is monic of degree k.

   (b) Obtaining a_0 is simple. We have $P_A(0) = a_0$, and, as $P_A(X) = \det(XI - A)$, this means that $a_0 = P_A(0) = \det(0I - A) = \det(-A)$. To obtain a_{m-1}, we use induction. For a 1×1 matrix, it can be seen to be $-\operatorname{tr}(A)$. We assume that this holds for matrices up to $m \times m$. Now, from before, we know that $P_A(X) = (X - a_{11})\det(XI - A_{11}) + \sum_{i=1}^{m}(-1)^{i+1}a_{1i}\Delta_{1i}$. However, none of these tail terms contribute to a_{m-1}, as their degree is too low. It is the a_{m-1} term of $(X - a_{11})\det(XI - A_{11})$. By induction hypothesis, $\det(XI - A_{11})$ leads with $X^{m-1} - \operatorname{tr}(A_{11})X^{m-2}$. Multiplying through, we get that $a_{m-1} = -a_{11} - \operatorname{tr}(A_{11}) = -\operatorname{tr}(A)$, as desired.

(c) Let A be a matrix. Then $P_A(X) = \det(XI - A) = \det(XI - A)^\tau = \det(XI^\tau - A^\tau) = \det(XI - A^\tau) = P_{A^\tau}(X)$.

(d) Let S invertible and A arbitrary. Then $P_{SAS^{-1}}(X) = \det(XI - SAS^{-1}) = \det(XSIS^{-1} - SAS^{-1}) = \det(S(XI - A)S^{-1}) = \det(XI - A) = P_A(X)$.

5. (a) The characteristic polynomial is $x^4 - 1$, and so we get four eigenvalues, $1, -1, i, -i$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ -i \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix}$$

when working over a field where -1 has square roots $\pm i$. If -1 is not square, then there are only two eigenvalues and eigenvectors.

(b) The characteristic polynomial is $x^4 - 16x^3 + 90x^2 - 216x + 189 = (x-3)^3(x-7)$. So it has four eigenvalues, $3, 3, 3, 7$, with eigenvectors

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

which works over any field at all.

(c) For this one, the formulas are very complicated, and can be obtained by careful calculation and application of the cubic formula.