1. (a) Assume that $G$ is abelian. Then let $(x, y), (x', y') \in G$. Then $(x, y)(x', y') = (x', y')(x, y)$. Using the group law, this is $(xx', yy') = (x'x, y'x)$, which means that $xx' = x'x$ and $yy' = y'y$, and so $G_1, G_2$ are both abelian. Conversely, assume that $G_1, G_2$ are abelian. Let $(x, y), (x', y') \in G$. Then $(x, y)(x', y') = (xx', yy')$. As $G_1, G_2$ are abelian, this is just $(x'x, y'y) = (x', y')(x, y)$, and so $G$ is abelian.

(b) This is false, as we can take $G = \mathbb{R}$, $x = 1$, and $y = 2$. If $(x, y) = (1, 2)$, then $(x, y)(1, 2) = (1, 4)$, which is not equal to $(1, 2)(1, 2) = (1, 4)$.

4. This is false, as if $R_1, R_2$ are skew fields, look at the elements of $R = R_1 \times R_2$ given by $(1_{R_1}, 0_{R_2})$ and $(0_{R_1}, 1_{R_2})$. Neither of these is zero in $R$. However, their product is $(0_{R_1}, 0_{R_2}) = 0_R$, and so they are zero divisors. This implies that they cannot be units, and so not every element of $R$ is invertible.

5. (a) For this whole part of the problem, let $f, g, h \in \mathcal{F}(X, G)$ and $x \in X$ be arbitrary.

Closure: By definition, $(f * g)(x) = f(x)g(x)$. As $G$ is a group, $f(x)g(x)$ is an element of $G$. Thus, $f * g$ defines a function by $x \mapsto f(x)g(x)$ which takes elements of $X$ to elements of $G$, and so $f * g \in \mathcal{F}(X, G)$.

Associativity: We start with $(f * g) * h(x)$. This is equal to $(f(x)g(x))h(x)$, by the definition of $*$. Now, as $G$ is associative, this is equal to $f(x)(g(x)h(x))$. By the definition of $*$ again, we can write this as $(f * (g * h))(x)$, and so $(f * g) * h = f * (g * h)$.

Identity: Define the function $e : X \to G$ by taking every $x \in X$ to $e_G$, the identity in $G$. Now, we look at $(f * e)(x) = f(x)e(x) = f(x)e_G = f(x) = e_Gf(x) = e(x)f(x) = (e * f)(x)$. Thus, $e$ is the identity for the operation $*$.

Inverses: Define the function $f_{inv}$ by for all $x \in X$, $f_{inv}(x) = f(x)^{-1}$. We claim that this is the inverse function for $f$ under $*$. We have $(f_{inv} * f)(x) = f_{inv}(x)f(x) = f(x)^{-1}f(x) = e_G = f(x)f(x)^{-1} = f(x)f_{inv}(x) = (f * f_{inv})(x)$, which shows that this is the case.

Commutativity: Assume that $G$ is an abelian group. Then $(f * g)(x) = f(x)g(x) = g(x)f(x) = (g * f)(x)$, and so $*$ is commutative.

Now assume that $*$ is commutative. For $a \in G$, we define $f_a \in \mathcal{F}(X, G)$.
\[ F(X, G) \] to be the function taking all \( x \in X \) to \( a \in G \). Let \( a, b \in G \). Then \( ab = f_a(x)f_b(x) = (f_a * f_b)(x) = f_{ab}(x) = f_b(x)f_a(x) = ba \), and so \( G \) is abelian.

(b) Let \( f, g \in F(X, G) \) and let \( y \in Y \subset X \). Then \( \phi(f * g)(y) = (f * g)(y) \) by the definition of \( \phi \). This is then \( f(y)g(y) \), and then by the definition of \( \phi \) again, we have \( f(y) = \phi(f)(y) \) and \( g(y) = \phi(g)(y) \), and so have \( f(y)g(y) = \phi(f)(y)\phi(g)(y) \). Then by the definition of \( \ast \) on \( Y \), we have \( \phi(f * \phi(g))(y) \), and so, putting it all together, we have that \( \phi(f * g)(y) = (\phi(f) * \phi(g))(y) \), and so \( \phi \) is a homomorphism.

(c) The image of \( \phi \) consists of all functions \( f : Y \to G \) which can be obtained by restricting a function \( \tilde{f} : X \to G \). Fix \( f : Y \to G \) arbitrary. Then define \( \tilde{f} \) by \( \tilde{f}(x) = \begin{cases} f(x) & x \in Y, \\ e_G & x \notin Y. \end{cases} \) For all \( y \in Y \), we have \( f(y) = \tilde{f}(y) \), and so \( \phi(\tilde{f}) = f \). Thus, \( \tilde{f} \) is in the image of \( \phi \). As \( \tilde{f} \) was arbitrary, this shows that \( \phi \) is surjective, and so \( \text{Im}(\phi) = F(Y, G) \).

The kernel of \( \phi \) consists of those functions \( f : X \to G \) which have \( \phi(f) \) equal to the function \( e_Y : Y \to G \) which maps all \( y \in Y \) to \( e_G \). This will be precisely the functions \( f : X \to G \) which have \( f(y) = e_G \) for all \( y \in Y \), by the definition of \( \phi \).

6 (a) Assume that \( H < G \). Then we want to show that \( F(X, H) < F(X, G) \). Now, for any group, a subset is a subgroup if and only if for all \( x, y \) in the subset, \( xy^{-1} \) is in it as well. Let \( f, g \in F(X, H) \). Look at \( f * g_{inv} \). Fix \( x \in X \). Then \( (f * g_{inv})(x) = f(x)g_{inv}(x) = f(x)g(x)^{-1} \), and as \( f(x) \) and \( g(x) \) are in \( H \), which is a subgroup, \( f(x)g(x)^{-1} \in H \), and so \( f * g_{inv} \) defines a map \( X \to H \), thus telling us that \( F(X, H) \) is a subgroup of \( F(X, G) \).

Assume that \( F(X, H) \) is a subgroup of \( F(X, G) \). Let \( a, b \in H \). We want to show that \( ab^{-1} \in H \). Define \( f_a, f_b : X \to H \) by \( f_a(x) = a \) and \( f_b(x) = b \) for all \( x \in X \). Then \( f_a, f_b \in F(X, H) \). Thus, in particular, \( f_a * f_b \in F(X, H) \). Now, evaluate on \( x \), and we obtain \( (f_a * f_b)(x) = f_a(x)f_b(x) = f_a(x)f_b(x)^{-1} = ab^{-1} \). Now, as \( f_a * f_b \in F(X, H) \), anything we get by evaluation is in \( H \), and so \( ab^{-1} \in H \). So \( H \) is a subgroup of \( G \).

(b) Assume that \( H \) is normal in \( G \). We want to show that for all \( g \in F(X, G) \) and all \( h \in F(X, H) \), we have \( g * h * g_{inv} \in H \). To do so, let \( g \in F(X, G) \), \( h \in F(X, H) \) and \( x \in X \) arbitrary. Then look at \( (g * f * g_{inv}) \). We must show that this maps into \( H \). Evaluating on \( x \), we obtain \( (g * f * g_{inv})(x) = g(x)f(x)g(x)^{-1} \). Now, as \( f(x) \in H \) and \( g(x) \in G \), with \( H \) normal in \( G \), we have \( g(x)f(x)g(x)^{-1} \in H \), as desired.

Now, assume that \( F(X, H) \) is normal in \( F(X, G) \). Keep our definition of \( f_a \) from above for all \( a \in G \). Let \( g \in G \) and \( h \in H \). We claim that
First, we will show that if $f$ functions $x$ a normal subgroup, we have that $f$ is normal in $G$. Evaluating on $x$, we get the element of $H$ \( f(x) = f_g(x)f_h(x)f_{g,h}(x)^{-1} = ghg^{-1} \in H \), so $H$ is normal in $G$.

(c) Assume that $H$ is a normal subgroup of $G$ (and thus, by the previous part, $\mathcal{F}(X,H)$ is a normal subgroup of $\mathcal{F}(X,G)$). Then we claim that there exists $\phi : \mathcal{F}(X,G/H) \rightarrow \mathcal{F}(X,G)/\mathcal{F}(X,H)$ which is an isomorphism of groups.

We define $\phi$ as follows: let $f : X \rightarrow G/H$ be any function. Then for each $x$, $f(x) = gH$ for some $g \in G$. We set $\phi(f)$ to be the class of functions $f': X \rightarrow G$ which contains the function $f'(x) = g$, with $g$ as above. We must prove that $\phi$ is well-defined, a homomorphism, injective and surjective.

To see that it is well defined, let $f : X \rightarrow G/H$, let $x \in X$ and let $g, g' \in G$ such that $f(x) = gH = g'H$. Then there are two candidates for $\phi(f)$, functions with $f'(x) = g$ and $f''(x) = g'$ (defined point by point on $X$). We must show that they are in the same class modulo $\mathcal{F}(X,H)$. To do so, we must show that $f' \ast f''^{-1}$ is in the same class as the identity map, because then $f'$ and $f''$ have the same inverse class. Now, $(f' \ast f''^{-1})(x) = f'(x)f''(x)^{-1} = gg'^{-1}$. Now, $gg'^{-1}$ is an element of $H$, as $gH = g'H = Hg' = Hg$ (with the last because $H$ is normal), and we can right multiply by $g'^{-1}$ to obtain the equation $H = Hgg'^{-1} = gg'^{-1}H$. Thus, there exists a function $\alpha : X \rightarrow H$ such that $f' \ast f''^{-1} = \alpha$, and so $f'$ and $f''$ define the same class. So $\phi$ is a well defined function.

Next we must check that it is a homomorphism. Let $f, g : X \rightarrow G/H$. Then $f \ast g$ is a function $X \rightarrow G/H$. So $\phi(f \ast g)$ defines a class in the quotient $\mathcal{F}(X,G)/\mathcal{F}(X,H)$. Let $x \in X$. Set $f(x) = aH$ and $g(x) = bH$. Then $(f \ast g)(x) = abH$. Now, apply $\phi$ to $f \ast g$ and we get $\phi(f \ast g)(x)$ is a function $\alpha : X \rightarrow G$ such that $\alpha(x) = ab$. We want to show that $\alpha$ is in the same class as $\phi(f) \ast \phi(g)$.

Now, $(\phi(f) \ast \phi(g))(x) = \phi(f(x))\phi(g(x)) = ab$, and so for all $x \in X$, we have $\phi(f)(x)\phi(g)(x) = \alpha(x) = \phi(f \ast g)(x)$, and so $\phi$ is a homomorphism.

Now we show that it is injective. Let $f : X \rightarrow G/H$ such that $\phi(f)$ is a function $X \rightarrow H$. Let $x \in X$, then $f(x) = gH$ for some $g \in G$, and $\phi(f)(x) = g$. Now, $\phi(f)$ is a function $X \rightarrow H$ if and only if $g \in H$, and so $f(x) = H$ in the first place, which is the same thing as $f$ being the identity element in $\mathcal{F}(X,G/H)$, so the kernel is trivial, and $\phi$ is injective.

Finally, we want to show that $\phi$ is surjective. Let $f : X \rightarrow G$ by any function. Define $\tilde{f} : X \rightarrow G/H$ by for all $x \in X$, $\tilde{f}(x) = f(x)H$. Then $\phi(\tilde{f})(x) = f(x)$ for all $x \in X$, and so $\phi(\tilde{f}) = f$. Thus, $\phi$ is surjective, and so is an isomorphism.

8 First, we will show that if $|X| > 1$ then $\mathcal{F}(X,R)$ cannot be a skew field,
and then we will show that, if \(|X| = 1\), then \(\mathcal{F}(X, R)\) is a skew field if and only if \(R\) is, which solves the problem.

Assume that \(|X| > 1\). Let \(x, y \in X\) with \(x \neq y\). Then define \(f \in \mathcal{F}(X, R)\) by \(f(x) = 1_R\) and \(f(a) = 0\) for all \(a \in X\) with \(a \neq x\) and define \(g(y) = 1_R\) and \(g(a) = 0\) for all \(a \in X\) not equal to \(a\). Then for all \(a \in X\), we have \((f * g)(a) = f(a)g(a) = 0\), because at least one of them is zero. Thus, \(\mathcal{F}(X, R)\) has zero divisors and cannot be a skew field.

Now assume that \(|X| = 1\). Denote the single element of \(X\) by \(x\). We will in fact prove a stronger statement: \(\mathcal{F}(X, R)\) is isomorphic to \(R\), regardless of the properties of \(R\). Define a map \(\gamma : \mathcal{F}(X, R) \rightarrow R\) by for each \(f : X \rightarrow R\), \(\gamma(f) = f(x)\). We first prove that this is a homomorphism of rings with identity.

Let \(f, g \in \mathcal{F}(X, R)\). Then \(\gamma(f \oplus g) = (f \oplus g)(x) = f(x) + g(x) = \gamma(f) + \gamma(g)\) and \(\gamma(f \odot g) = (f \odot g)(x) = f(x)g(x) = \gamma(f)\gamma(g)\), so it is a ring homomorphism. Additionally, if \(e\) is the multiplicative identity, then it must have \(e(x) = 1_R\), and so \(\gamma(e) = 1_R\), so it preserves the identity. Now, we must show surjective and injective.

To see that \(\gamma\) is surjective, let \(r \in R\). Then we can define a function \(f : X \rightarrow R\) by \(f(x) = r\) for the only \(x \in X\). Then, \(\gamma(f) = f(x) = r\), and so \(\gamma\) is surjective. To see that it is injective, let \(f\) be any function such that \(\gamma(f) = 0\). Then \(f(x) = 0\), but the only element of \(X\) is \(x\), so \(f\) is the constant function mapping all of \(X\) to zero, which is the additive identity of \(\mathcal{F}(X, R)\). Thus, the kernel is trivial and \(\gamma\) is injective, and so is an isomorphism.

As \(\mathcal{F}(X, R)\) is isomorphic to \(R\), certainly \(R\) is a skew field if and only if \(\mathcal{F}(X, R)\) is.