Homework 9 Solutions

2 From discussion in recitation, $\mathscr{L}((R^2)^2, R)$ is the dual of $R^2 \otimes_R R^2$. So it suffices to show that $R^2 \otimes_R R^2$ has dimension 4. By definition, it has a basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$, and so $\mathscr{L}((R^2)^2, R)$ has dimension four. The other two cases are trickier. Let $\phi \in \mathscr{L}^s((R^2)^2, R)$. Then $\phi(m,n) = \phi(n,m)$. So, $\tilde{\phi}(\sum a_{ij} e_i \otimes e_j)$ must have $a_{ij} = a_{ji}$, and any map with this condition is symmetric. Thus, $\phi(ae_1 + be_2, ce_1 + de_2) = ac\phi(e_1, e_1) + (ad + bc)\phi(e_1, e_2) + bd\phi(e_2, e_2)$. Thus, the space of symmetric maps is dual to the submodule of $R^2 \otimes_R R^2$ generated by $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_2$, and thus has dimension three. So now, we must determine what submodule of $R^2 \otimes_R R^2$ the space $\mathscr{L}^{alt}((R^2)^2, R)$ is dual to. For an alternating map ϕ, we have $\phi(ae_1 + be_1, ce_1 + de_2) = ac\phi(e_1, e_1) + ad\phi(e_1, e_2) + bc(\phi(e_2, e_1) + db\phi(e_2, e_2)$, but also that $\phi(e_1, e_1) = \phi(e_2, e_2) = 0$ and $\phi(e_1, e_2) = -\phi(e_2, e_1)$. Thus, $\phi(ae_1 + be_2, ce_1 + de_2) = (ad - bc)\phi(e_1, e_2)$, and so $\mathscr{L}^{alt}((R^2)^2, R)$ is dual to the span on $e_1 \otimes e_2$, and so is one dimensional, as desired.

3 We must show that this set is linearly independent and spanning. First, spanning. Let $f : T \to R$ any function. Set $a_i = f(x_i)$, for some ordering of $T = \{x_1, \ldots, x_n\}$. Then look at $\sum a_i f_{x_i}$. We claim that this is f. Applying it to x_j, we obtain $\sum a_i f_{x_i}(x_j) = \sum a_i \delta_{ij} = a_j = f(x_j)$, and so $f = \sum a_i f_{x_i}$. Now, linear independence. Let $f = \sum a_i f_{x_i} = 0$. Then, $f(x_j) = 0$, and so $\sum a_i f_{x_i}(x_j) = \sum a_i \delta_{ij} = a_j = 0$, but this must hold for all j, and so all the coefficients must be 0. Thus, we have a spanning set.

3 Let A be an integer matrix whose inverse, A^{-1}, is also an integer matrix. Then $AA^{-1} = I$. Taking determinants, we have $\det(AA^{-1}) = \det I = 1$. But $\det AA^{-1} = \det A \det A^{-1}$, and so $\det A \det A^{-1} = 1$. Thus, $\det A$ is a unit in $\mathbb{Z}$, and so must be ± 1.

6 [a]

Let A, B, C as in the problem. Then we note that for any matrix, there's a larger field such that it is equivalent to an upper triangular matrix, and that determinant isn't changed by changing between equivalent matrices. Let A', B' be upper triangular matrices equivalent to A, B. Then, the whole matrix $\begin{pmatrix} A' & C \\ 0 & B' \end{pmatrix}$ is upper triangular, and has determinant $\det A' \det B' = \det A \det B$. To see that this is the determinant of $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, we note that we can get from this matrix to the one we want by using a change of basis block matrix.

(d) For this, the same trick suffices, and you can turn each A_i into an upper triangular matrix without affecting any determinants.

17 By Cramer's Rule, $x_i = \Delta_i/\Delta$, so we will merely compute Δ_i and Δ and declare solutions.

a

$$\Delta = \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -1 \end{pmatrix} = 2$$

$$\Delta_1 = \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 0 & -1 & -1 \end{pmatrix} = 1,$$

$$\Delta_2 = \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 1 & 0 & -1 \end{pmatrix} = 4,$$

$$\Delta_3 = \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ -1 & -1 & 0 \end{pmatrix} = -3$$

and so $x = 1/2, y = 2, z = -3/2$.

b Similarly, here,

$$\Delta = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 5 \\ 1 & 1 & 5 & 6 \end{pmatrix} = -1,$$

$$\Delta_1 = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \\ 1 & 1 & 4 & 5 \\ 0 & 1 & 5 & 6 \end{pmatrix} = -4,$$

$$\Delta_2 = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 4 \\ 1 & 1 & 4 & 5 \\ 1 & 0 & 5 & 6 \end{pmatrix} = 2,$$

$$\Delta_3 = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 4 \\ 1 & 1 & 1 & 5 \\ 1 & 1 & 0 & 6 \end{pmatrix} = 4,$$

$$\Delta_4 = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 5 & 0 \end{pmatrix} = -3,$$

so $x = 4, y = -2, z = -4, w = -3$