Complex manifolds, Kahler metrics, differential and harmonic forms

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1 Lecture 1

Definition 1.1 (Complex Manifold). A complex manifold is a manifold with coordinates holomorphic on $\mathbb{C}^n$ rather than $\mathbb{C}^\infty$ on $\mathbb{R}^n$.

What is the difference between holomorphic and $\mathbb{C}^\infty$?

From the PDE point of view, they must satisfy the Cauchy-Riemann equations: $f(z) = u + iv$ is holomorphic if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The fact that this causes the function $f$ to be analytic (holomorphic) is what gives the theory a very algebraic flavor.

Example 1.2 (Sphere). We get a map $S^2 \setminus \{N\} \to \mathbb{C}$ and another $S^2 \setminus \{S\} \to \mathbb{C}$ by stereographic projection. The map from $S^2 \setminus \{N, S\}$ to itself that takes the image of the first to the image of the second is $\frac{1}{z}$ on $\mathbb{C}^*$, which is holomorphic. So what are the holomorphic functions? Any holomorphic function gives a bounded entire function by removing a point, and so must be constant! This is in fact true for holomorphic functions on compact connected complex manifolds.

Theorem 1.3. The only global holomorphic functions on a compact connected complex manifold are the constants.

Corollary 1.4. There are no compact complex submanifolds on $\mathbb{C}^n$ of dimension greater than 0.

Example 1.5 (Projective Space). Set $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\} / \sim$ where $z' \sim z$ iff there exists $\lambda \in \mathbb{C}^*$ such that $z' = \lambda z$. We put charts on it by looking at $U_i = \{|z| \in \mathbb{P}^n | z_i \neq 0\}$.

Question 1. When is a compact complex manifold a submanifold of $\mathbb{P}^n$?

We will also be interested in

Example 1.6 (Grassmannians). $G(k, n)$ is the space of $k$-dimensional subspaces of $\mathbb{C}^n$.

Two manifolds may be $\mathbb{C}^\infty$ diffeomorphic but not holomorphic.
Example 1.7 (Torus). As a $C^\infty$ manifold, a torus is just $S^1 \times S^1$. However, it can have many complex structures. To get one, view it as $\mathbb{C}/\Lambda$ where $\Lambda$ is a lattice isomorphic to $\mathbb{Z}^2$. There are many different lattices, and so we can get many complex structures this way. It is not hard to see that they are distinct.

The theory of complex manifolds splits into compact and noncompact, we’re only going to look at the compact complex manifolds.

So what do we do? We have no holomorphic functions. A function can be viewed as being a map $M \to M \times \mathbb{C}$ by $x \mapsto (x, f(x))$. To get something like a function, we can change $M \times \mathbb{C}$ to something that is only locally this product, that is, a line bundle.

Definition 1.8 (Line Bundle). A holomorphic line bundle is $E \xrightarrow{\pi} M$ where $M$ and $E$ are complex manifolds and $\pi$ is holomorphic such that the fiber over each point is isomorphic to $\mathbb{C}$ and it is locally biholomorphic to $U_\alpha \times \mathbb{C}$.

Taking a trivializing open cover $U_\alpha$, we get maps $U_\alpha \cap U_\beta \to GL(1, \mathbb{C})$ (or $n$ in the case of more general vector bundles) which we will call $g_{\alpha\beta}$, and they will satisfy $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$, these are called the gluing data.

We can get another bundle using $g_{\alpha\beta}^{-1}$, and these will give the dual bundle, whose fibers are naturally the dual of the original, and call it $E^*$. Similarly, we can take $E, F$ and put the transition functions in blocks, and we get $E \oplus F$.

For line bundles, the transition functions are numbers, not matrices, so their product is another invertible function, so we get a line bundle $E \otimes F$.

Example 1.9 (Universal Line over $\mathbb{P}^n$). We set $T = \{(z, v)|v \in [z]\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ and its projection map to $\mathbb{P}^n$ makes it a line bundle with transition maps $z_i/z_j$.

So, we replace the notion of a global function with a global section of a line bundle $E \to M$, which is a map $M \to E$ such that $x \in E_x$. This means that on each $U_\alpha$ we have a function $f_\alpha$ and $f_\alpha = g_{\alpha\beta}f_\beta$.

Let $P(x_0, \ldots, z_n)$ be a homogeneous polynomial of degree $k$. Then on $U_i$, set $f_i = P/z_i^k$, these form a global section of $(T^*)^k$. But, if we remove the dual, there are no global sections.

2 Lecture 2

Example 1.16 and A.1

In mathematics, we often try to reduce problems to linear algebra, because linear algebra is something that we understand very well.

If we set $g_{\alpha\beta} = D(\phi_\alpha \circ \phi_\beta^{-1}) \in GL(n, \mathbb{R})$. In fact, we get \[
\begin{pmatrix}
A & -B \\
B & A
\end{pmatrix} \in GL(2n, \mathbb{R}),
\]
and these are the changes of coordinates preserving the matrix \[
\begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}.
\]

On a complex manifold $M$, it is possible to define a linear map $J_p : T_p M \to T_p M$ by $J_p(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$ and $J_p(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$. 

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Then $J^2_p = -id$ and so $J^2 + I = 0$.

Now, take $[T_p M]_C = T_p M \otimes \mathbb{R} \mathbb{C} = T_p M \oplus iT_p M$. Call $T^\prime_p$ and $T^\prime\prime_p$, the parts with eigenvalue $i$ and $-i$.

We can actually, for a complex manifold $M$, make $T_p M$ into a complex vector space, by defining $(a + ib) \ast v = av + bJv$. And then, this vector space is isomorphic to $T^\prime_p M$ by taking $v$ to $v - iJv$.

So what is the tangent bundle of $\mathbb{P}^n$? Let $L(t) \in \mathbb{P}^n$ a curve. Fix $v \in L(p) = L$. So the curves don’t depend on motion inside the line, thus we end up with $T(\mathbb{P}^n) = \text{hom}_\mathbb{C}(T, \mathbb{C}^{n+1}/T)$, and by changing the lines to subspaces, the same is true of Grassmannians.

Now, we write $[T^\prime_p M]_C = T^{1,0}_p(M) \oplus T^{0,1}_p(M)$ where the first has basis $dz_i = dx_i + idy_i$ and the second has basis $d\bar{z}_i = dx_i - idy_i$. Then, $df = \sum_j df(\frac{\partial}{\partial z_j})d\bar{z}_j + \sum_j df(\frac{\partial}{\partial \bar{z}_j})dz_j$.

We now set $A^{p,q}(M)$ to be the space of forms whose terms have $dz_i \wedge d\bar{z}_j$ where $|I| = p$ and $|J| = q$. Then the exterior derivative maps $d : A^{p,q}(M) \rightarrow A^{p+1,q}(M) \oplus A^{p,q+1}(M)$. So $d = \partial + \bar{\partial}$, both of which square to zero, and also $\partial\bar{\partial} = 0$.

We then have that $A^k(M, \mathbb{C}) = \oplus_{p+q=k} A^{p,q}(M)$. The Dolbeault cohomology, then, is $H^{p,q}_{\bar{\partial}}(M)$ is the cohomology of $\bar{\partial}$.

Inside of $A^{p,0}(M)$ is $\Omega^p(M)$, which is the collection of the forms with holomorphic, not merely $C^\infty$, coefficients.

Now we note that the Dolbealt cohomology is NOT diffeomorphism invariant. It depends on the complex structure of the manifold in question.

## Lecture 3

Let $V$ be a real vector space along with an operator $J^2 = -I$. This makes it a complex vector space. We can also say $V_C = V \oplus iV = V' \oplus V''$ where $V'$ is the $i$-eigenspace and $V''$ the $-i$-eigenspace. We write $V_C^* = V^1 \oplus V^{0,1}$ along with $J^*$, which are the $i$ and $-i$ eigenspaces. Then $\wedge^r V_C^* = \bigoplus \wedge^{p,q}$ with $p + q = r$. So $\wedge^2 V_C^* = \wedge^2, \wedge^1, \wedge^0$, which are $-1, 1, -1$ eigenspaces respectively.

So now, we have $d : A^{p,q}(M) \rightarrow A^{p+1,q} \oplus A^{p,q+1}$ and for each $p$ we get a complex $0 \rightarrow \Omega^p(U) \rightarrow A^{p,0}(U) \rightarrow A^{p,1} \rightarrow \ldots$ which is exact for a small enough $U$ (more precisely, exact as a complex of sheaves).

For $p = n = \dim M$, then $\Omega^n(U) = f dz_1 \wedge \ldots \wedge dz_n$ with $f$ holomorphic.

**Proposition 3.1.** The Following are equivalent

1. A symmetric bilinear form $B : V \times V \rightarrow \mathbb{R}$ such that $B(Ju, Jv) = B(u, v)$
2. An alternating form $\omega : V \times V \rightarrow \mathbb{R}$ such that $\omega(Ju, Jv) = \omega(u, v)$
3. A hermitian form $H : (V, J) \times (V, J) \rightarrow \mathbb{C}$ with $H(Jv, Ju) = H(v, u)$.

Now, we move to manifolds. Every complex manifold has a positive definite Hermitian structure on the holomorphic tangent bundle, which is equivalent to every complex manifold has a Riemannian metric compatible with $J$. 

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By this, we mean that on \((U, z_1, \ldots, z_n)\), we have that \(h_{jk} = H(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k})\) and \(\omega = \frac{i}{2} \sum_{j,k} h_{jk} d\bar{z}_j \wedge dz_k\).

We define a hermitian structure on \(M\) to be Kahler if \(d\omega = 0\). This implies that \(h_{jk}(z) = \delta_{jk} + O(|z|^2)\).

### 3.1 Symplectic and Kähler Manifolds

A symplectic manifold is a pair \((M, \omega)\) where \(d\omega = 0\) and \(\omega^n \neq 0\), with \(\omega\) a 2-form. We’ll assume that \(M\) is compact. Then \(M\) being Kähler implies that \(M\) is symplectic, because \(\int_M \omega^n \neq 0\), and so \(0 \neq [\omega]^n \in H^{2n}(M)\), so each \([\omega^k]\) is nonzero.

**Example 3.2.** Calabi and Eckmann proved that for \(n, m \geq 1\), there was a complex structure on \(S^{2n+1} \times S^{2m+1}\), and these can never be Kähler.

In fact, any compact symplectic manifold has an almost complex structure.

**Example 3.3.** \(\mathbb{C}^n\) is Kähler with metric \(\frac{i}{2} \sum_{i,j} \delta_{ij} dz_i \wedge d\bar{z}_j\).

**Example 3.4.** \(\mathbb{P}^n\) has a Kähler structure, by taking on each \(U_j\) the function \(\rho_j(|z|) = \sum z_j^2 > 0\). On \(U_j \cap U_k\), we have \(|z_j|^2 \rho_j = |z_k|^2 \rho_k\), we then take logs and apply \(\partial \bar{\partial}\), and we find that \(\partial \bar{\partial} \log(\rho_j) = \partial \bar{\partial} \log(\rho_k)\). So we set \(\omega_j = -\frac{1}{2\pi i} \partial \bar{\partial} \log(\rho_j(z))\), and the metric we construct is the Fubini-Study metric.

Let \(N \subset M\). Then for all \(p \in N\), there exist coordinates \((U, z_1, \ldots, z_n)\) on \(M\) around \(p\) such that \(N \cap U\) is described by \(z_1 = \ldots = z_k = 0\). If \(M\) is Kähler and \(N \subset M\) is a submanifold, then \(N\) is Kähler.

Thus, if \(M\) is a submanifold of \(\mathbb{P}^n\), then \(M\) is Kähler. Thus, \([\omega] \in H^2(M) \cap H^2(M, \mathbb{Z})\) is necessary.

### 4 Lecture 4

Let’s look at the real, smooth case. Let \(M\) be a compact oriented Riemannian manifold.

If \(V\) is a real vector space which is oriented with an inner product, then \(\bigwedge^k(V^*)\) has an inner product as well.

**Exercise 4.1.** Show that \(\langle \alpha_1 \wedge \ldots \wedge \alpha_r, \beta_1 \wedge \ldots \wedge \beta_r \rangle = \det(\langle \alpha_i, \beta_j \rangle)\).

Volume element \(\Omega \in \bigwedge^n(V^*)\) given by \(\xi_1 \wedge \ldots \wedge \xi_n\), where the \(\xi_i\) form an orthonormal basis.

We have a map \(*: \bigwedge^k(V^*) \to \bigwedge^{n-k}(V^*)\), and \(*(\xi_1 \wedge \ldots \wedge \xi_r) = \text{sign}(I, J)\xi_j\wedge \ldots \wedge \xi_{j-r}\). So then \(\alpha \wedge *\beta = \langle \alpha, \beta \rangle \Omega\).

Now, \(*\) is an isomorphism, and it satisfies \(*^2 = (-1)^{(n-r)}\).

Back to the manifold \(M\). We can define an inner product on forms \(\alpha, \beta\) by \(\int_M \alpha \wedge *\beta\). This is a positive definite bilinear form.
Now, we define $\delta = (-1)^{nr+1} d\ast$, and it takes $r$-forms to $r-1$-forms, using this $\ast$ operator. We claim that $d$ and $\delta$ are adjoints, that is, $(d\alpha, \beta) = (\alpha, \delta\beta)$.

$\int_M d\alpha \wedge \ast \beta = \int_M d(\alpha \wedge \ast \beta) - (-1)^r \alpha \wedge d\ast \beta$, but this will just be $-(-1)^r \int_M \alpha \wedge d\ast \beta$, up to sign, we can just insert a $\ast$ in front of $d$, and the signs work out.

Now, if $d\alpha = 0$, then $\ast \alpha$ may not be closed, but it is if and only if $\delta \alpha = 0$.

**Proposition 4.2.** \(d\alpha = \delta\alpha = 0\) if and only if \((d\delta + \delta d)\alpha = 0\).

One direction is simple, for the other, we have \(0 = \langle (d\delta + \delta d)\alpha, \alpha \rangle\), which using the adjoint property proves the result.

We call this operator \(\Delta\), the Laplace-Beltrami operator, or the Laplacian, and we call any form \(\alpha\) with \(\Delta \alpha = 0\) a harmonic form.

**Exercise 4.3.** $\ast \Delta = \Delta \ast$.

Why should we hope that every cohomology class has a harmonic form in it?

Heuristically, start with $d\alpha = 0$. Then \([\alpha]\) is the set of forms of the form $\alpha + d\beta$. Then $|\alpha + t d\beta|^2 = \langle \alpha, \alpha \rangle + 2t \langle \alpha, d\beta \rangle + t^2 |d\beta|^2$. Now, suppose $|\alpha|^2$ is a maximum. Then for all $\beta$ we’ll have that $\langle \alpha, d\beta \rangle = 0$ and $\langle \delta\alpha, \beta \rangle = 0$, so we can assume that $\delta\alpha = 0$.

**Theorem 4.4** (Hodge’s Theorem). 1. $\mathcal{H}^k(M)$, the harmonic forms, is finite dimensional

2. $A^r(M) = \mathcal{H}^k \oplus \Delta(\mathcal{H}^r(M))$.

In particular, every form is a harmonic form, plus $d$ of something plus $\delta$ of something. So then any form is $\alpha + d\beta + \delta\gamma$. But if it’s closed, then $d\delta\gamma = 0$, which implies that $\delta\gamma = 0$, so for any closed form, it is of the form $\alpha = \eta + d\beta$.

Thus, $H^k_{dR}(M, \mathbb{R}) \cong \mathcal{H}^k(M)$.

Take a submanifold $Z^{n-k} \subset M^n$, then for any form in $H^{n-k}(M)$, we can restrict it to $Z$ and integrate to get a map to $\mathbb{R}$. By Poincaré duality, this gives us a class $\eta_Z \in H^k(M)$.

Now, we define everything in a “hermitian way.” So we take $\int_M \alpha \wedge \ast \bar{\beta}$, and note that $\ast$ takes $(p, q)$ to $(n-q, n-p)$.

We define $\partial^* = - \ast d\ast$ and $\bar{\partial}^* = - \ast \bar{\partial} \ast$, and these are of type $(0, -1)$ and $(-1, 0)$ and match with $\partial$ and $\bar{\partial}$. So then we have $\Delta_{\beta} = \partial \bar{\partial}^* + \bar{\partial}^* \partial$, and we can write any form as a sum $\alpha = \alpha^{k,0} + \alpha^{k-1,1} + \ldots$

And, we leave off with the fact that, on a Kähler manifold $\Delta_d = 2\Delta_{\beta}$.

5 Lecture 5

We’ll be working on a compact Kähler manifold. We know that $\Delta = 2\Delta_{\bar{\partial}}$.

**Corollary 5.1.** If $\alpha \in A^k(M)$ is harmonic and $\alpha = \sum \alpha_{p,q}$, then $\Delta \alpha_{p,q} = 0$.

Let $H^{p,q}(M)$ be the set of classes in $H^{p+q}(M)$ such that $\alpha$ has a representation in bidegree $p, q$. 

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Theorem 5.2. \( H^k_{dR}(M, \mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(M) \) and \( H^{p,q}(M) \cong \mathcal{H}^{p,q}(M) = H^{p,q}_0(M) \cong H^q(M, \Omega^p) \).

We in fact have \( H^{p,q}(M) = \overline{H^{p,q}(M)} \).

Now, if \( k \) is odd, we have \( H^k(M) = H^{k,0} \oplus \ldots \oplus H^{0,k} \), and there is no middle term, so the dimension of \( H^k(M) \) is even when \( k \) is odd.

The cup product will actually respect the bigrading: \( H^{p,q} \cup H^{p',q'} \subset H^{p+p',q+q'} \). Voisin has found examples where every condition is satisfied except this property.

Why should all of these nice things be true? Let \( \omega \in H^2(M, \mathbb{R}) \cap H^{1,1}(M) \). We have a Lefschetz map \( L_\omega : \Lambda^* \to \Lambda^* \) by \( \alpha \mapsto \omega \wedge \alpha \) increasing degree by \( (1,1) \). We have \( L_{n+1} = 0 \). Now, we can define \( Y : \Lambda^* \to \Lambda^* \) by \( Y(\alpha) = (n-k)\alpha \) for \( \alpha \in \Lambda^k \), and we have \( [Y, L] = -2L \). Now, define \( N_+ = (-1)^k \ast L \ast \) on \( \Lambda^* \).

This is the adjoint of \( L \), and it satisfies \( [Y, N_+] = 2N_+ \) and \([N_+, L] = Y \), so this actually gives us a representation of the Lie algebra \( \mathfrak{sl}_2 \! \! \! \)!

In this representation, \( L \) and \( N_+ \) are shifts, and the eigenvalues of \( Y \) are the degrees of forms, \( L \) increases degree and \( N_+ \) decreases. Then there’s the Lefschetz Theorem which says, first, that \( L_{n} : \Lambda^{n-k}(T_p) \to \Lambda^{n+k}(T_p) \) is an isomorphism. Moreover, it tells us that there are two types of cohomology classes \( \Lambda^{n-k}(T_p) = P^{n-k} \oplus L(\Lambda^{n-k-2}) \). We call this the Lefschetz decomposition for forms.

We in fact have the following relations:


These imply that \( [\Delta, L] = [\Delta, Y] = [\Delta, N_+] = 0 \), by just writing it out and putting \( L \) between the partials. This implies that the Lefschetz theorem holds if we replace forms by cohomology classes. We call this the Hard Lefschetz Theorem.

This gives another topological restriction: the betti numbers (even and odd separately) must be increasing to the middle degree.

Now, let us assume \( \dim_{\mathbb{C}} M = 1 \), so we are on a Riemann surface. So then \( H^1 = H^{1,0} \oplus H^{0,1} \), and by the Dolbeaut theorem, \( H^{1,0} \cong H^{1,0}_{dR} \cong H^0(M, \Omega^1) \), the holomorphic 1-forms, that can be written \( f(z)dz \) locally, for \( f \) holomorphic. Then there is \( H^{0,0} \) and \( H^{1,1} \), so the Hodge numbers are easy: \( h^{0,0} = h^{1,1} = 1 \) and \( h^{1,0} = h^{0,1} = g \).

On a complex surface, things are slightly more interesting, and \( H^{1,1} \) splits into \( H^{1,1}_0 + \mathbb{C}\omega \), where subscript of zero will indicate the primitive cohomology.