Hodge Loci and Absolute Hodge Classes

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1 Charles 1 - Hodge Loci and Absolute Hodge Classes

Let $X$ be a compact Kähler manifold. We have Hodge classes $H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X)$.

**Conjecture 1.1** (Hodge Conjecture). If $X$ is projective, Hodge classes are algebraic.

**Theorem 1.2** (Voisin). This is false for Kähler.

Now, let $k$ be a field, embeddable in $\mathbb{C}$. For $X$ smooth and projective over $k$, we have $H^i_{dR}(X/k)$. Now, take $\sigma : k \to \mathbb{C}$ an embedding.

We get Betti-deRham isomorphisms which are compatible with the cycle class map.

**Definition 1.3** (Relative Hodge Class). Let $\alpha \in H^{2i}_{dR}(X/k)$. We say that $\alpha$ is a Hodge class relative to $\sigma$ is $\alpha \in F^i H^{2i}_{dR}(X/k)$ and the image of $\alpha$ in $H^{2i}(X_{\sigma}(\mathbb{C}), \mathbb{Q}(i)) \otimes \mathbb{C}$ lies in the rational subspace.

We say that $\alpha$ is absolute Hodge if it is a Hodge class relative to any $\sigma$.

Remarks: First, $X/\mathbb{C}$, we can define what it means for a class to be an absolute Hodge class. Now, how dependent is this on $k$? The cohomology classes of algebraic cycles are absolute Hodge.

Proof: If $Z$ is an algebraic cycle in $X$, then for any $\sigma : k \to \mathbb{C}$, we have $Z \times_\sigma \mathbb{C}$ algebraic cycle in $X \times_\sigma \mathbb{C}$ So this gives a Hodge class relative to $\sigma$, and works for all $\sigma$.

Thus, we have two conjectures:

**Conjecture 1.4.** Hodge classes are absolute.

**Conjecture 1.5.** Absolute Hodge classes are algebraic.

These two conjectures imply the Hodge conjecture.

Let $X/\mathbb{C}$ be smooth and projective of dimension $d$. then we have $H^d(X \times X) = \oplus_i H^i(X) \otimes H^{d-i}(X)$ contains a class $[\Delta] = \sum \pi_i$ via the Künneth formula.

It’s a conjecture that the $\pi_i$ are algebraic.
**Proposition 1.6.** The $\pi_i$ are absolute Hodge cycles.

Conjecture: The inverse of the Lefschetz map is algebraic.

**Proposition 1.7.** This is absolute.

**Theorem 1.8** (Deligne). Hodge classes on abelian varieties are absolute.

Let $X/\mathbb{C}$ be smooth and projective, $Z$ an algebraic cycle of codimension $i$ in $X$, homologically equivalent to zero. We have $AJ(Z) \in J^i(X)$, which gives us an extension of mixed Hodge structures $0 \to H^{2n-1}(X,\mathbb{Z}(n)) \to H \to \mathbb{Z} \to 0$.

Now, $AJ(Z) = 0$ if and only if the sequence splits which is if and only if there exists a Hodge class $H$ mapped to 1 in $\mathbb{Z}$.

This is related to the Bloch-Beilinson filtration on Chow Groups, $F^2CH^i(X) = \ker AJ$.

Now, we have our GM connection and a VHS structure.

Remark: If $X/k$ and $k = \bar{k}$, and $K \supset k$, then $CH^i(X)_{\bar{k}} = CH^i(X_k)$. But, this does not hold for cohomological equivalence!

Indeed, take $Z \subset X_K$ be an algebraic cycle. Then the cohomology class $[Z]_{dR} \in F^iH^{2i}(X/K)_{\bar{k}}$ actually comes from $F^iH^{2i}(X/k)$.

**Proposition 1.9.** For $\alpha \in F^iH^{2i}(X_K/K)$ an absolute Hodge cycle, then $\alpha$ is defined over $k$.

**Proof.** We can assume that $K$ is finitely generated over $k$, and $K$ is the function field of some quasiprojective $S/k$. We can assume also that $\alpha$ extends to $\bar{\alpha}$, a section of $F^iH^{2i}_{dR}(X/k) \otimes \mathcal{O}_S$. We want to prove that $\alpha$ is constant.

Now, pull everything back to $\mathbb{C}$ by some $k \subset \mathbb{C}$. We want to check that $\alpha_{\mathbb{C}}$ is constant.

But, $\alpha$ being absolute, for every $k \to \mathbb{C}$, $\alpha_\sigma$ is going to lie the rational lattice of $H^{2i}_{dR}(X/\mathbb{C}) \otimes \mathcal{O}_S$, and so will be constant.

**Theorem 1.10** (Principle B of Deligne). $\pi : \mathcal{X} \to S$ a smooth projective morphism with $S$ quasi-projective and connected. Let $\alpha$ be a global section of $R^2\pi_*\mathbb{Q}(i)$, which is Hodge everywhere. Then if $\alpha_\sigma$ is absolute Hodge for some $\sigma \in S(\mathbb{C})$, it is for all $\sigma \in S$.

**Proof.** Just conjugate $\alpha$. $\nabla = \nabla^\sigma$ for $\sigma \in \text{Aut}\mathcal{C}$ Then $\nabla \alpha^\sigma = \nabla^\sigma \alpha^\sigma = (\nabla \alpha)^\sigma = 0$. 

Now, we need $\alpha$ algebraic as a section of $\mathcal{H}_{\mathcal{X}/S}$. For this, we take the global invariant cycle theorem which says that $\alpha$ is the restriction of a Hodge class in $H^{2i}(Y,\mathbb{Q}(-1))$ where $Y \supset \mathcal{X}$ is a smooth compactification.

Application: Let $H$ be a HS of weight 2 with $h^{2,0} = 1$. Then there exists $H'$ a Hodge structure of weight 1, polarized, with $H \to \text{End}H'$.

Starting with $\pi : \mathcal{X} \to S$ a family of polarized K3’s, get $\alpha : \mathcal{A} \to S$ a polarized abelian scheme, then we have $R^2\pi_*\mathbb{Q}(1) \to \text{End}(R^1\alpha_*\mathbb{Q})$ for all $S$. connection.
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Let $X/k$ be a smooth projective (maybe quasi-projective) variety and $k$ of characteristic 0 and embeddable into $\mathbb{C}$.

**Definition 2.1.** Fix $\sigma : k \to \mathbb{C}$ and $\alpha \in H_{\text{dR}}^{2i}(X/k)$, say that $\alpha$ is Hodge with respect to $\sigma$ if with the isomorphism $H_{\text{dR}}^{2i}(X/k) \otimes \mathbb{C} \cong H_{\text{B}}(X\mathbb{C}(\mathbb{C}), \mathbb{Q}(i)) \otimes \mathbb{C}$ then $\alpha$ lies in the rational subspace and $\alpha \in F^iH_{\text{dR}}^{2i}(X/k)$.

We say that $\alpha$ is absolute Hodge if $\alpha$ is Hodge with respect to every $\sigma$.

Now, if $X/\mathbb{C}$ and $\alpha \in H_{\text{B}}^{2i}(X(\mathbb{C}), \mathbb{Q}(i))$ is absolute Hodge if is is a hodge class and its image in $H_{\text{dR}}^{2i}(X/\mathbb{C})$ is absolutely Hodge, so Hodge with respect to all $\sigma \in \text{Aut}(\mathbb{C})$.

**Theorem 2.2** (Principle B). Let $\pi : X \to S$ be smooth, projective over $\mathbb{C}$ with $S$ quasiprojective, smooth and connected. Let $\alpha$ be a global section of $R^{2i}\pi_*\mathbb{Q}(i)$ such that $\alpha$ is of type $(i,i)$ at any point of $S$ and for some $s_0 \in S(\mathbb{C})$, $\alpha_{s_0}$ is absolutely Hodge. Then $\forall s \in S(\mathbb{C})$, we have that $S(\mathbb{C})$ is absolutely Hodge.

Remark: If we know $\alpha_{s_0}$ is algebraic, then we can prove $\alpha_s$ is.

We prove this before.

**Theorem 2.3** (Global Invariant Cycle Theorem). Let $\pi : X \to S$ as above, $\rho : \pi_1(S(\mathbb{C}), s) \to \text{Aut}(H^{2i}(X_s, \mathbb{Q}(i)))$ the monodromy representation. Then the fixed points of $\rho$ are those in the image of $H^{2i}(Y_s, \mathbb{Q}(i)) \to H^{2i}(X_s, \mathbb{Q}(i))$ where $Y \supset X$ is a smooth compactification.

In our proof, $\alpha$ is a global section of $R^{2i}\pi_*\mathbb{Q}(i)$, and so it is $\pi_1$-invariant and thus comes from $Y$, so is algebraic. Then $\forall s \in S(\mathbb{C})$, we have that $\alpha_s$ is absolute Hodge.

Remark: if $X$ is defined over $\mathbb{Q}$, then Hodge classes on $X$ are always absolute Hodge.

Question: How can we reduce questions in Hodge theory (Hodge cycles are absolute, Hodge conjecture) to questions over $\mathbb{Q}$ and number fields?

Let $X/\mathbb{C}$ be smooth projective. We can always find $X \to S/\mathbb{Q}$ with $\pi$ smooth and projective, $S$ smooth and quasiprojective and $s \in S(\mathbb{C})$ such that $X \cong X_s$ ($X \subset \mathbb{P}^n_\mathbb{C}$ corresponds to a point of some Hilbert-scheme of subschemes of $\mathbb{P}^n_\mathbb{C}$).

What does it mean for Hodge classes fibers to be absolute?

Remark: $X \to S/\mathbb{Q}$, so the Hodge bundle $\mathcal{H}_{\text{dR}}^i(X/S)$ are defined over $\mathbb{Q}$, and so is $\nabla$.

If $\sigma \in \text{Aut}(\mathbb{C})$, and $\sigma$ permutes the complex points of the Hodge bundle, then

**Proposition 2.4.** $\alpha \in F^iH_{\text{dR}}^{2i}(X_s/\mathbb{C})$ is an absolute Hodge class if and only if $\alpha$ is rational and $\sigma(\alpha)$ is an absolute Hodge class for all $\sigma \in \text{Aut}(\mathbb{C})$.

**Definition 2.5** (Hodge Locus). The locus of Hodge classes for $\pi$ is the set of $\alpha_t \in H_{\text{dR}}^{2i}(X_t/\mathbb{C})$ such that $\alpha_t$ is a hodge class.
Proposition 2.6. Hodge classes for the fibers of $\pi$ are absolute iff the locus of Hodge classes is invariant under $\text{Aut}(\mathbb{C})$.

The locus of Hodge classes is actually a countable union of analytic subvarieties of $\mathcal{H}^{2i}(X/S)$.

Start with $\alpha_t \in H^{2i}_{\text{dR}}(X_s/\mathbb{C})$. We get a component of the locus of Hodge classes by taking parallel transport of $\alpha_s$ over some small open subset and looking at the vanishing on $\mathcal{H}^{2i}/F^i$.

Lemma 2.7. Let $B$ be defined over $\mathbb{Q}$ and $Z \subset B(\mathbb{C})$ be a countable union of analytic subvarieties such that $Z(\mathbb{C})$ is stable under the action of $\text{Aut}(\mathbb{C})$. Then $Z$ is a countable union of algebraic varieties over $\mathbb{Q}$.

The idea is to take a very general point of $Z$ and look at the orbit of it under $\text{Aut}(\mathbb{C})$.

We know the geometric part of the conclusion:

Theorem 2.8 (Deligne-Cattani-Kaplan). Let $\pi : \mathcal{X} \to S$ as before. Then the locus of Hodge classes in $\mathcal{H}^{2i}(\mathcal{X}/S)$ is a countable union of algebraic subvarieties.

We don’t get information on the field of definition.

Theorem 2.9. Let $\mathcal{X} \to S/\mathbb{Q}$ as before. The Hodge locus of $\pi$ is the image in $S$ of the component of the locus of Hodge classes passing through $\alpha$. By DCK, this locus is algebraic. Let $Z_\alpha$ be in this locus. Assume it is defined over $\overline{\mathbb{Q}}$. Then the Hodge conjecture for $(\mathcal{X}, Z_\alpha)$ can be reduced to the HC for some $\mathcal{X}/\overline{\mathbb{Q}}$.

Proof follows from DCK and the global invariant cycle theorem.