Hodge Loci and Absolute Hodge Classes

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1 Charles 1 - Hodge Loci and Absolute Hodge Classes

Let X be a compact Kähler manifold. We have Hodge classes $H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X)$.

Conjecture 1.1 (Hodge Conjecture). *If X is projective, Hodge classes are algebraic.*

Theorem 1.2 (Voisin). *This is false for Kähler.*

Now, let k be a field, embeddable in $\mathbb{C}$. For X smooth and projective over k, we have $H^i_{dR}(X/k)$. Now, take $\sigma : k \to \mathbb{C}$ an embedding.

We get Betti-deRham isomorphisms which are compatible with the cycle class map.

Definition 1.3 (Relative Hodge Class). *Let $\alpha \in H^{2i}_{dR}(X/k)$. We say that α is a Hodge class relative to σ is $\alpha \in F^i H^{2i}_{dR}(X/k)$ and the image of α in $H^{2i}(X_\sigma(\mathbb{C}), \mathbb{Q}(i)) \otimes \mathbb{C}$ lies in the reational subspace.*

We say that α is absolute Hodge if it is a Hodge class relative to any σ.

Remarks: First, $X/\mathbb{C}$, we can define what it menas for a class to be an absolute Hodge class. Now, how dependent is this on k? The cohomology classes of algebraic cycles are absolute Hodge.

Proof: If Z is an algebraic cycle in X, then for any $\sigma : k \to \mathbb{C}$, we have $Z \times_\sigma \mathbb{C}$ algebraic sycle in $X \times_\sigma \mathbb{C}$ So this gives a Hodge class relative to σ, and works for all σ.

Thus, we have two conjectures:

Conjecture 1.4. *Hodge classes are absolute.*

Conjecture 1.5. *Absolute Hodge classes are algebraic.*

These two conjectures imply the Hodge conjecture.

Let $X/\mathbb{C}$ be smooth and projective of dimension d. then we have $H^d(X \times X) = \oplus_i H^i(X) \otimes H^{d-i}(X)$ contains a class $[\Delta] = \sum \pi_i$ via the Künneth formula.

It's a conjecture that the π_i are algebraic.

Proposition 1.6. *The π_i are absolute Hodge cycles.*

Conjecture: The inverse of the Lefschetz map is algebraic.

Proposition 1.7. *This is absolute.*

Theorem 1.8 (Deligne)**.** *Hodge classes on abelian varieties are absolute.*

Let $X/\mathbb{C}$ be smooth and projective, Z an algebraic cycle of codimension i in X, homologically equivalent to zero. We have $AJ(Z) \in J^i(X)$, which gives us an extension of mixed Hodge structures $0 \to H^{2n-1}(X, \mathbb{Z}(n)) \to H \to \mathbb{Z} \to 0$.

Now, $AJ(Z) = 0$ if and onyl if the sequence splits which is if and only if there exists a Hodge class H mapped to 1 in $\mathbb{Z}$.

This is related to the Bloch-Beilinson filtration on Chow Groups, $F^2CH^i(X) = \ker AJ$.

Now, we have out GM connection and a VHS structure.

Remark: If X/k and $k = \bar{k}$, and $K \supset k$, then $CH^i(X)K) \neq CH^i(X_k)$. But, this does not for cohomological equivalence!

Indeed, take $Z \subset X_K$ be an algebraic cycle. Then the cohomology class $[Z]_{dR} \in F^iH^{2i}(X/K)$ actually comes from $F^iH^{2i}(X/k)$.

Proposition 1.9. *For $\alpha \in F^iH^{2i}(X_K/K)$ an absolute Hodge cycle, then α is defined over k.*

Proof. We can assume that K is finitely generated over k, and K is the function field of some quasiprojective S/k. We can assume also that α extends to $\bar{\alpha}$, a section of $F^1H^{2i}_{dR}(X/k) \otimes \mathscr{O}_S$. We want to prove that α is constant.

Now, pull everything back to $\mathbb{C}$ by some $k \subset \mathbb{C}$. We want to check that $\alpha_{\mathbb{C}}$ is constant.

But, α being absolute, for every $k \to \mathbb{C}$, α_σ is going to lie the rational lattice of $H^i_{dR}(X/\mathbb{C}) \otimes \mathscr{O}_S$, and so will be constant. $\square$

Theorem 1.10 (Principle B of Deligne)**.** *$\pi : \mathcal{X} \to S$ a smooth projective morphism with S quasi-projective and connected. Let α be a global section of $R^{2i}\pi_*\mathbb{Q}(i)$, which is Hodge everywhere. Then if α_s is absolute Hodge for some $s \in S(\mathbb{C})$, it is for all $s \in S$.*

Proof. Just conjugate α. $\nabla = \nabla^\sigma$ for $\sigma \in \mathrm{Aut}\mathbb{C}$ Then $\nabla\alpha^\sigma = \nabla^\sigma\alpha^\sigma = (\nabla\alpha)^\sigma = 0$. $\square$

Now, we need α algebraic as a section of $\mathscr{H}_{\mathcal{X}/S}$. For this, we take the global invariant cycle theorem which says that α is the restriction of a Hodge class in $H^{2i}(Y, \mathbb{Q}(-1))$ where $\mathcal{Y} \supset \mathcal{X}$ is a smooth compactification.

Application: Let H be a HS of weight 2 with $h^{2,0} = 1$. Then there exists H' a Hodge structure of weight 1, polarized, with $H \to \mathrm{End}H'$

Starting wtih $\pi : \mathcal{X} \to S$ a family of polarized K3's, get $\alpha : \mathscr{A} \to S$ a polarized abelian scheme, then we have $R^2\pi_*\mathbb{Q}(1) \to End(R^1\alpha_*\mathbb{Q})$ for all S. connection.

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Let X/k be a smooth projective (maybe quasi-projective) variety and k of characteristic 0 and embeddable into $\mathbb{C}$.

Definition 2.1. *Fix $\sigma : k \to \mathbb{C}$ and $\alpha \in H^{2i}_{dR}(X/k)$, say that α is Hodge with respect to σ if with the isomorphism $H^{2i}_{dR}(X/k) \otimes \mathbb{C} \cong H_B(X_{\mathbb{C}}(\mathbb{C}), \mathbb{Q}(i)) \otimes \mathbb{C}$ then α lies in the rational subspace and $\alpha \in F^i H^{2i}_{dR}(X/k)$.*

We say that α is absolute Hodge if α is Hodge with respect to every σ.

Now, if $X/\mathbb{C}$ and $\alpha \in H^{2i}_B(X(\mathbb{C}), \mathbb{Q}(i))$ is absolute Hodge if is is a hodge class and its image in $H^{2i}_{dR}(X/\mathbb{C})$ is absolutely Hodge, so Hodge with respect to all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Theorem 2.2 (Principle B)**.** *Let $\pi : \mathcal{X} \to S$ be smooth, projective over $\mathbb{C}$ with S quasiprojective, smooth and connected. Let α be a global section of $R^{2i}\pi_*\mathbb{Q}(i)$ such that α is of type (i,i) at any point of S and for some $s_0 \in S(\mathbb{C})$, α_{s_0} is absolutely Hodge. Then $\forall s \in S(\mathbb{C})$, we have that $S(\mathbb{C})$ is absolutely Hodge.*

Remark: If we know α_{s_0} is algebraic, then we can prove α_s is.

We prove this before.

Theorem 2.3 (Global Invariant Cycle Theorem)**.** *Let $\pi : \mathcal{X} \to S$ as above, $\rho : \pi_1(S(\mathbb{C}), s) \to \mathrm{Aut}(H^{2i}(X_s, \mathbb{Q}(i))$ the monodromy representation. Then the fixed points of ρ are those in the image of $H^{2i}(\mathcal{Y}, \mathbb{Q}(i)) \to H^{2i}(X_s, \mathbb{Q}(i))$ where $\mathcal{Y} \supset \mathcal{X}$ is a smooth compactification.*

In our proof, α is a global section of $R^{2i}\pi_*\mathbb{Q}(i)$, and so it is π_1-invariant and thus comes from $\mathcal{Y}$, so is algebraic. Then $\forall s \in S(\mathbb{C})$, we have that α_s is absolute Hodge.

Remark: if X is defined over $\mathbb{Q}$, then Hodge classes on X are always absolute Hodge.

Question: How can we reduce questions in Hodge theory (Hodge cycles are absolute, Hodge conjecture) to questions over $\mathbb{Q}$ and number fields?

Let $X/\mathbb{C}$ be smooth projective. We can always find $\mathcal{X} \xrightarrow{\pi} S/\mathbb{Q}$ with π smooth and projective, S smooth and quasiprojective and $s \in S(\mathbb{C})$ such that $X \cong X_s$ ($X \subset \mathbb{P}^n_{\mathbb{C}}$ corresponds to a point of some Hilbert-scheme of subschemes of $\mathbb{P}^n_{\mathbb{C}}$)

What does it mean for Hodge classes fibers to be absolute?

Remark: $\mathcal{X} \to S/\mathbb{Q}$, so the Hodge bundle $\mathscr{H}^i_{dR}(\mathcal{X}/S)$ are defined over $\mathbb{Q}$, and so is ∇.

If $\sigma \in \mathrm{Aut}(\mathbb{C})$, and σ permutes the complex points of the Hodge bundle, then

Proposition 2.4. *$\alpha \in F^i H^{2i}_{dR}(X_s/\mathbb{C})$ is an absolute Hodge class if and only if α is rational and $\sigma(\alpha)$ is an absolute Hodge class for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.*

Definition 2.5 (Hodge Locus)**.** *The locus of Hodge classes for π is the set of $\alpha_t \in H^{2i}_{dR}(X_t/\mathbb{C})$ such that α_t is a hodge class.*

Proposition 2.6. *Hodge classes for the fibers of π are absolute iff the locus of Hodge classes is invariant under* $\mathrm{Aut}(\mathbb{C})$.

The locus of Hodge clases is actually a countable union of analytic subvarieties of $\mathscr{H}^{2i}(\mathcal{X}/S)$.

Start with $\alpha_t \in H^{2i}_{dR}(X_s/\mathbb{C})$. We get a component of the locus of Hodge classes by taking parallel transport of α_s over some small open subset and looking at the vanishing on $\mathscr{H}^{2i}/F^i$.

Lemma 2.7. *Let B be defined over $\mathbb{Q}$ and $Z \subset B(\mathbb{C})$ be a countable union of analytic subvarieties such that $Z(\mathbb{C})$ is stable under the action of* $\mathrm{Aut}(\mathbb{C})$. *Then Z is a countable union of algebraic varieties over $\mathbb{Q}$.*

The idea is to take a very general point of Z and look at the orbit of it under $\mathrm{Aut}(\mathbb{C})$.

We know the geometric part of the conclusion:

Theorem 2.8 (Deligne-Cattani-Kaplan)**.** *Let $\pi : \mathcal{X} \to S$ as before. Then the locus of Hodge classes in $\mathscr{H}^{2i}(\mathcal{X}/S)$ is a countable union of algebraic subvarieties.*

We don't get information on the field of definition.

Theorem 2.9. *Let $\mathcal{X} \to S/\mathbb{Q}$ as before. The Hodge locus of π is the image in S of the component of the locus of Hodge classes passing through α. By DCK, this locus is algebraic. Let Z_α be in this locus. Assume it is defined over $\bar{\mathbb{Q}}$. Then the Hodge conjecture for (X, Z_α) can be reduced to the HC for some $\mathcal{X}/\bar{\mathbb{Q}}$.*

Proof follows from DCK and the global invariant cycle theorem.