Applications to the Beilinson-Bloch Conjecture

Green

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1 Green 1 - Applications to the Beilinson-Bloch Conjecture

California is like Italy without the art. - Oscar Wilde

Let $X$ be a smooth projective variety. There are two ways to look at it. One is to look at it as a compact Kähler manifold with a Hodge metric giving a projective embedding. The other is to look at $X$ as already a subset of $\mathbb{P}^n(k)$ given by explicit equations and work algebraically.

We use the Hodge metric to get a Hodge structure, which sits inside $\Gamma\setminus D$. A lot of what you get here comes from algebra, but aren’t Hodge structures of varieties, and even occasionally from analysis.

We can take the field of definition of $X$, $k$, to be finitely generated over $\mathbb{Q}$, by noting that $X$ is defined over $\mathbb{Q}$ adjoin the coefficients of the equations. Now, some things we can do with $k$ as an abstract field, but others, we can do only when $k \subset \mathbb{C}$.

Let’s look at $\mathbb{Q}(\pi)$. As $\pi$ is transcendental, we can represent elements of this field by $p(\pi)/q(\pi)$ where $p, q \in \mathbb{Q}[x]$. We can similarly describe $\mathbb{Q}(\epsilon)$. Abstractly, these fields are isomorphic. But they’re different subfields of $\mathbb{C}$. We can think of these as both being $\mathbb{Q}(x)$, which is distinct from $\mathbb{Q}(\sqrt{7})$.

So look at the elliptic curves $y^2 = x(x - 1)(x - \pi)$ and $y^2 = x(x - 1)(x - \epsilon)$. As abstract curves over $\mathbb{Q}(x)$, they’re isomorphic. However, over $\mathbb{C}$, the Hodge structures are not equivalent.

In mathematics, we’ve got a break between technology, which we need to prove theorems, and intuition, which we need to figure out theorems.

If we write $k$ as a field finitely generated over $\mathbb{Q}$, then $k = \mathbb{Q}(\alpha_1, \ldots, \alpha_t)[\beta_1, \ldots, \beta_r]$ where $\alpha_1, \ldots, \alpha_t$ are algebraically independent and $\beta_1, \ldots, \beta_r$ are algebraic over it.

Enter geometry. $\mathbb{Q}(x) = \mathbb{Q}(\mathbb{P}^1)$, so $k = \mathbb{Q}(S)$ for $S$ defined over $\mathbb{Q}$, and $\mathbb{Q}(S_1) \cong \mathbb{Q}(S_2)$ if $S_1$ and $S_2$ are birational over $\mathbb{Q}$.

A point $s \in S(\mathbb{C})$ is a geometric point, or a very general point of $S$ if $s$ does not lie on any proper $\mathbb{Q}$-subvariety of $S$ (that is, the Zariski closure of $s$ over $\mathbb{Q}$ is $S$)

Example 1.1. For $S = \mathbb{P}^1$, $s$ is very general if and only if $s$ is transcendental.
Now, let \( k = \mathbb{Q}(\alpha) \) for \( \alpha \) transcendental, and look at \( y^2 = x(x-1)(x-\alpha) \). For every very general point of \( S \), in fact, for \( s \neq 0,1,\infty \), we get a smooth elliptic curve. So we have \( X' \to S \setminus E \), and \( E \), the discriminant locus, is defined over \( \mathbb{Q} \).

**Example 1.2.** Let \( k \) be a number field. Then \( k = \mathbb{Q}(\alpha) \) and \( \alpha \) has minimal polynomial \( p(x) \in \mathbb{Q}[x] \). Now let \( S \) be the variety defined by \( p \), that is, a finite number of points. \( S \) is defined over \( \mathbb{Q} \), but the points are not, individually, defined over \( \mathbb{Q} \). \( X \), defined over \( k \), means that we get \( [k : \mathbb{Q}] \) complex varieties, one for each embedding into \( \mathbb{C} \).

**Example 1.3.** Let \( X \) be defined over \( \mathbb{Q} \) and \( p \in X \) a very general point. Then \( X \to \mathbb{P}^n(\mathbb{Q}) \) by \( p \mapsto (p_0, \ldots, p_n) \) then \( k = \mathbb{Q}(p_1/p_0, \ldots, p_n/p_0) \). We can take \( S = X \), so transcendence degree is \( \dim X \).

**Example 1.4.** Let \( X \) be defined over \( \mathbb{Q} \). Take \( (p,q) \in X \times X \) a very general point. Then we get an embedding into \( \mathbb{P}^n \), and the field’s transcendence degree is \( 2 \dim X \).

**Example 1.5.** Look at hypersurfaces of degree \( d \) in \( \mathbb{P}^n \). If \( F = \sum_{|I|=d} a_I x^I \), then if we look at \( k = \mathbb{Q}(a_I) \) we get a much larger dimensional projective space for \( S \), and we have \( X' \to S \) the universal family of hypersurfaces.

So, which computations do we actually need the complex embeddings for? Grothendieck learned to compute cohomology groups using just \( k \), but for the Hodge structures, really need \( \mathbb{C} \).

Big Point: Hodge structures require a complex embedding.

Set \( y^2 = x(x-1)(x-\alpha) = f(x) \). If we differentiate, we get \( 2ydy = f'(x)dx \), so we have \( \frac{2dy}{f(x)} = \frac{dx}{y} \), and this lets us represent the holomorphic 1-form on \( E \) as a power series and integrate.

Now, let \( \lambda \) be a simple closed homotopically nontrivial curve in \( E \). Look at \( \int_\lambda \omega \). We can write \( \lambda = \sigma_1 + \ldots + \sigma_n \) a bunch of lines, and so we need to integrate \( \omega \) along each of these, and \( \int_\sigma \frac{dx}{y} = \int_{\sigma} \frac{dx}{\sqrt{f(x)}} \), and we need \( \int_C \frac{da}{\sqrt{x(x-1)(x-\alpha)}} \).

We can expand as a power series and integrate.

Now, the point is that the integral lattice \( H^*(X,\mathbb{Z}) \subset H^*(X,\mathbb{C}) \) is going to depend transcendently on \( s \), on complex embedding \( k \to \mathbb{C} \).

Now, take \( X \) a smooth variety defined over \( k \). We’ll be wanting \( \Omega^1_{X(k)/k} \) to be the Kähler differentials over \( k \). That is, the module \( d(f+g) = df + dg, \) \( da = 0 \) for \( a \in k \) and \( d(fg) = f dg + g df \). We also have \( \Omega^1_{X(k)/\mathbb{Q}} \) where we only have \( da = 0 \) for \( a \in \mathbb{Q} \).

Now, \( da = 0 \) for all \( a \in k \) actually implies \( da = 0 \) for all \( a \in \bar{k} \). Why? Look at the minimal polynomial of \( \alpha \), \( a_0\alpha^m + \ldots + a_m = 0 \). Take \( d \) of this, and the Liebniz rule implies that we have some element of \( k \) times \( da = 0 \), and so \( da = 0 \).

So \( \Omega^1_{k/\mathbb{Q}} \) is a \( k \) vector space with basis \( da_1, \ldots, da_\ell \).

These Kähler differentials give us a complex \( \Omega^*_{X(k)/\mathbb{Q}} \), and need to be careful, this complex doesn’t end at \( \dim X \).
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Let $X$ be a smooth variety defined over $k$ where $k$ is finitely generated over $\mathbb{Q}$. Then we defined last time $\Omega^1_{X(k)/k}$ and $\Omega^1_{X(k)/\mathbb{Q}}$, and defined the complex $\Omega^*_X(k)/k$.

Now, we need to define hypercohomology. Start with a complex $A^*$, then $H^r(A^*) = \ker d / \text{im} d$ on $A^*$.

Now, look at $A^*\cdot$ a double complex, and take the total complex. This is a complex if $d, \delta$ the horizontal and vertical operators anticommute, and then we use $D = d + \delta$, and hypercohomology is defined by $\mathbb{H}^r(A^*\cdot) = H^r(T^*, D)$.

Note: Hypercohomology is not on the list in "My Favorite Things".

Using Cech cohomology, we define $A^{p,q} = C^q(\mathcal{U}, \Omega^p)$, and add a sign to the Cech differential, then we can take Hypercohomology.

Now, there’s a theorem that tells us that for $i_s : k \subset \mathbb{C}$, $\dim_k \mathbb{H}^n(\Omega^*_X(k)/k) = \dim_{\mathbb{C}} H^R_{dR}(X_s, \mathbb{C})$.

Theorem 2.1 (Grothendieck Comparison Theorem).

1. $\mathbb{H}^n(\Omega^*_X(k)/k) \otimes i_s \mathbb{C} \cong H^n(X_s, \mathbb{C})$
2. $H^q(\Omega^*_X(k)/k) \otimes i_s \mathbb{C} \cong H^{p,q}(X_s)$
3. $\mathbb{H}^n(\Omega^*_X(k)/k) \otimes i_s \mathbb{C} \cong F^p H^n(X_s, \mathbb{C})$

We can’t read off the integral lattice, though.

$\text{Gr}^n \Omega^*_X(k)/\mathbb{Q}$ is defined to by $F^m \Omega^*/F^{m+1} \Omega^*$ and that’s $\Omega^m_{k/\mathbb{Q}} \otimes \Omega^{* - m}_{X(k)/k}$.

He compares the derivation of the long exact sequence on cohomology to $X$-rays...not good for you to see too many times.

Looking at spreads: take $\mathcal{X} \to S$, this gives a variation of Hodge structure over $S \setminus \Sigma$, the smooth locus. Looking at the Gauss-Manin connection $\nabla$, we have a complex $(\Omega^*_k/\mathbb{Q} \otimes \mathbb{H}^q(\Omega^*_X(k)/k, \nabla))$, and we can get Griffiths Transversality (also called the infinitesimal period relations)

Now, say that $k = \mathbb{Q}(x_1, \ldots, x_T)[y_1, \ldots, y_A]/(p_1, \ldots, p_B)$, then $\Omega^1_{k/\mathbb{Q}}$ is generated by $dx_1, \ldots, dx_A$.

Example 2.2. Recall the example $y^2 = x(x-1)(x-\alpha)$. If we differentiate, but don’t assume that $d\alpha = 0$, we get $2ydy = f'(x)dx - x(x-1)d\alpha$. And now, $\frac{2dy}{f'(x)} = \frac{dx}{y} - \frac{x(x-1)}{f'(x)y}d\alpha$. And this last term gives us problems lifting. Fortunately, the coboundary map is essentially a measure of how much lifting fails, so the last term can be thought of as an element of $\Omega^1_{k/\mathbb{Q}} \otimes H^1(\mathcal{O}_{X(k)})$, so the whole thing is $\nabla \omega$ for some 1-form.

Now we introduce $Z^p(X(k))$, the codimension $p$ cycles defined over $k$ and $CH^p(X(k))$ the cycles mod rational equivalence defined over $k$.

So now, we define $K_p(\mathcal{O}_{X(k)})$ to be the Quillen $p$th K-group for $\mathcal{O}_{X(k)}$. Now, in the $k$ Zariski topology, we have $CH^p(X(k)) \cong H^p(K_p(\mathcal{O}_{X}))$. Modulo torsion, it’s a theorem of Soulé that we can replace Quillen K-theory with Milnor K-theory.

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We also start with \( X = \text{alence on} \ k/3 \) Green 3 this to the Absolute Hodge conjecture. \( H \) which should take \( F \) to them as soon as they’re online.

\( \otimes \)

\( s \) depend on which \( J \)

\( \text{Now look at} \ Example 3.2. \)

\( \text{out the set of extension classes, we get} \)

\( 0, \) and we can construct splittings that respect the Hodge filtration, and split-

\( d \alpha \)

\( \text{Then we have} \ \Omega^\ast \)

\( \text{defined over} \ k \)

\( \text{no idea how to compute this} \ \text{Ext group in general.} \)

Today we’re going to look at cycles over \( k \) on a variety \( X \) defined over \( Q \).
Conjecture 4.1 (Deligne-Bloch-Beilinson). Let $X$ be defined over $\mathbb{Q}$. Then $CH^p(X(\mathbb{Q}))_\mathbb{Q}$ is captured by cycles classes and $\mathcal{A}_X^p \otimes \mathbb{Q}$.

That is, $F^2 = 0$.

The Conjecture in fact says that $F^m = 0$ for $m \geq$ the transcendence degree of $k$, plus two, for $X$ defined over $k$.

Now, if $X$ is defined over $\mathbb{Q}$ and $k$ a finitely generated extension of $\mathbb{Q}$, we can find $S$ with $k = \mathbb{Q}(S)$, and set $X = X \times S$, and for any cycle $Z \in Z^p(X(k))$ we can spread it to $Z \in Z^p(X \times S(\mathbb{Q}))$, and there will exist a $W \subset S$ a proper subvariety, defined over $\mathbb{Q}$ of lower dimension with $W \in Z^{p-1}(X \times W)$ such that $Z \rightarrow Z + W$.

If the conjecture on varieties over $\mathbb{Q}$ is ok, then $Z \cong 0$ in rational equivalence over $\mathbb{Q}$ for some $W$, so $[Z + W]$ is torsion, and so its Abel-Jacobi image is also zero after tensoring with $\mathbb{Q}$.

(I lost track of the lecture here)