Algebraic de Rham Cohomology and Betti Cohomology

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We're going to be talking about the arithmetic aspects of things. These are the "absolute Hodge classes" and fields of definition.

The basic insight is Grothendieck's comparison theorem. Let X be a smooth quasiprojective variety over $k \supset \mathbb{Q}$, and we have all of the various Kähler differentials.

Definition 0.1 (Algebraic deRham cohomology). *We define* $H^i_{dR}(X/k) = \mathbb{H}^i(\Omega^*_{X/k})$, *which is a* k*-vector space.*

Note that if $L \supset k$, then $X_L = X \times_k L$ then we have $H^i_{dR}(X_L/L) = H^i_{dr}(X/k) \otimes_k L$.

Theorem 0.2. *If* X *is defined over* $\mathbb{C}$ *then* $H^i_{dR}(X/\mathbb{C}) \cong H^i(X^{an}, \mathbb{C})$.

Under this isomorphism $\mathbb{H}^i(\Omega^{\geq p}) \cong F^p H^i(X^{an}, \mathbb{C})$.

Remark: This is true for any quasiprojective variety (only poles at infinity) and we have two structures on $H^i(X^{an}, \mathbb{C})$. One is a $\mathbb{Q}$-structure, $H^i(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ the "Betti" structure, and the other is $H^i_{dR}(X/k) \otimes_k \mathbb{C}$, the deRham structure.

0.1 Families

Let $f : X \to B$ be a smooth projective variety over $\mathbb{C}$. By Katz-Oda, the Gauss-Manin connection is algebraically defined, and so we have our variations of Hodge structure are algebraic, and we have the SES

$$0 \to f^*\Omega^1_{B/k} \otimes \Omega^{*-1}_{X/B} \to \Omega^*_{X/B}/L^2\Omega^*_{X/B} \to \Omega^*_{X/B} \to 0$$

the connecting map gives the Gauss-Manin connection.

0.2 Cycle Classes and fields of definition

If X is a smooth projective variety over $\mathbb{C}$ and Z a subvariety of codimension p, then $[Z^{an}]_k \in H^{2p}(X^{an}, \mathbb{Q})$ is always a Hodge class.

An important point is that we can also define an algebraic fundamental class $[Z]_{dR} \in F^p H^{2p}_{dR}(X/k)$.

Theorem 0.3. *Let X, Z be defined over $\mathbb{C}$. Then $(2\pi i)^p[Z^{an}]_B = [Z]_{dR}$*

In general, use the chen classes of vector bundles with sections vanishing along Z.

(Detailed definition of chern classes)

Now, we can extend Chern classes to coherent sheaves by resoltuing using locally free sheaves (because X is smooth), and $c_p(\mathscr{O}_Z) = (-1)^{p-1}(p-1)![Z]$ for Z codimension p. And so, finally, we set $[Z]_{dR} = \frac{(-1)^{p-1}}{(p-1)!}c_p(\mathscr{O}_Z)$.