Algebraic varieties and schemes over any scheme.
Non singular varieties

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June 16, 2010

1 Lecture 1

Let $k$ be a field and $k[x_1, \ldots, x_n]$ the polynomial ring with coefficients in $k$. Then we have two objects: polynomials $P \in k[x_1, \ldots, x_n]$ and polynomial functions obtained by turning polynomials into maps $k^n \to k$. For finite fields, these are very different, though they are the same for infinite fields.

A subset $E \subset k^n$ is called an algebraic set if it is equal to the zero locus of a collection of polynomials.

**Theorem 1.1** (Finiteness Theorem of Hilbert). $k[x_1, \ldots, x_n]$ is Noetherian.

So any collection of polynomials has zero locus determined by finitely many of them.

1.1 Zariski Topology

Closed sets are the algebraic sets, $\emptyset$ and $k^n$, and open sets are the complements. This is actually non-Hausdorff.

**Theorem 1.2** (Hilbert Nullstellensatz). Let $k = \overline{k}$, then if $P$ vanishes on an algebraic set $E = V(I)$, then some power of $P$ is in the ideal $I$. In particular, this implies that $I(E) = \sqrt{I}$.

Let $E = \cup E_i$ where the $E_i$ are irreducible, that is, are not the union of two properly contained algebraic sets. Then, $E_i$ is irreducible if and only if $I(E_i)$ is prime.

If $f = P|_E$, we call it an algebraic function, and there are also regular functions.

**Theorem 1.3.** The set of algebraic functions on $E$ is isomorphic to $k[x_1, \ldots, x_n]/I(E) = A(E)$.

**Definition 1.4** (Regular function). $\phi$ is regular at $x \in E$ if there exists a Zariski neighborhood of $x$ such that $\phi = f/g$ on it.

$\phi$ is regular on $E$ if it is regular at each $x \in E$. 

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Theorem 1.5. The set of regular functions on $E$ is ring isomorphic to $A(E)$.

Definition 1.6 (Morphism). A morphism of varieties $\phi : E \to F$ is a continuous function which pulls regular functions back to regular functions $A(F) \to A(E)$.

1.2 Projective varieties

$\mathbb{P}_k^n = \mathbb{A}^{n+1} \backslash \{0\} / \sim$ where $x \sim y$ if there exists $\lambda \in k^*$ such that $x = \lambda y$. Now, we look at $R = k[x_0, \ldots, x_n]$ as a graded ring, with gradations the polynomials homogeneous of a given degree $d$.

Let $S$ be a set of homogeneous polynomials. Then $V(S)$ the algebraic projective set of zeroes. These give a Zariski topology on $\mathbb{P}_k^n$, and we see that $I(E)$ is a homogeneous ideal and we define the coordinate ring to be $R/I(E)$.

For projective algebraic sets, a regular function is one that is locally $f/g$ with $\deg f = \deg g$ and $g$ not vanishing on the neighborhood.

An algebraic variety is then a Zariski open subset of a projective variety, and this gives us a category of algebraic varieties.

1.3 Sheaves

Let $X$ be a topological space and look at the category of open sets. A presheaf is just a contravariant functor to the category of abelian groups.

A sheaf is a presheaf along with two conditions:

1. $s \in \mathcal{F}(U)$ such that its image in $\mathcal{F}(U_i)$ is zero for all $i$ is zero.
2. If we have $s_i \in \mathcal{F}(U_i)$ which agree when restricted to $U_i \cap U_j$, then there exists $s$ which restricts to each of them.

2 Lecture 2 - Sheaves

Let $X$ be a topological space, $\mathcal{F}, \mathcal{G}$ sheaves. A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is a map for each open set compatible with the restriction maps.

We define the stalk at $x \in X$ of $\mathcal{F}$ to be $\varprojlim_{U \ni x} \mathcal{F}(U)$ as abelian groups.

We define the associated sheaf to a given presheaf $\mathcal{F}$ to be the sheaf such that every map from the presheaf to any sheaf must factor through, and denote it $\tilde{\mathcal{F}}$. This is unique up to unique isomorphism.

The associated sheaf has the property that $\mathcal{F}_x \cong \tilde{\mathcal{F}}_x$ for all $x \in X$.

Now, let $f : X \to Y$ be a continuous map and $\mathcal{F}$ a sheaf on $X$. We define $f_* \mathcal{F}$ by $f_* \mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$.

For $Y \subset X$, and $\mathcal{F}$ on $X$, we define $f^{-1}(\mathcal{F})$ to be $\mathcal{F}|_Y$.

2.1 Ringed Spaces

A pair $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of rings on $X$ is a ringed space. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ and a map $\mathcal{O}_Y \to f_* \mathcal{O}_X$. 
A locally ringed space is a ringed space such that the stalks of the sheaf \( \mathcal{O}_X \) are all local rings, and a morphism of locally ringed spaces is required to induce on the stalks maps \( f^{-1}(m_{X,x}) = m_{Y,y} \).

Take \((E, \mathcal{O}_E)\) where \(E\) is an algebraic set and \(\mathcal{O}_E\) is the sheaf of regular functions. This example is the fundamental one in algebraic geometry.

The locally ringed space \((C^n, \mathcal{O}_{C^n})\) with the sheaf being the sheaf of local holomorphic functions is the fundamental example in analytic geometry.

### 2.2 Local Analytic Spaces

For \(U \subset C^n, f_i : U \to \mathbb{C}\) holomorphic for \(i \in I\). Then we know \((C^n, \mathcal{O}_{C^n})\) is a locally ringed space, look at \((U, \mathcal{O}_{C^n}|_U)\). Set \(X = \{f_i = 0, i \in I\}\) and then \((f_i)_{\mathcal{O}_U} = \mathcal{I}\) is a sheaf of ideals.

So, we can now distinguish between \((0, \mathbb{C})\) and \((0, \mathbb{C}\{x\}/x^2)\), the first is just a point, the second is a double point, and can be viewed as the intersection of a parabola and a line tangent to its vertex.

### 2.3 Affine Schemes

Let \(A\) be a ring (commutative with identity). Then Spec\(A\) is the set of prime ideals of \(A\), and the closed sets are given by taking an ideal \(I\) in \(A\) and setting \(V(I)\) to be the set of prime ideals containing \(I\). The open sets are their complements.

We define on Spec\(A\) the sheaf \(\mathcal{O}_A\), whose stalks at \(P\) is \(A_P\), and for any open set \(U\), we define \(\mathcal{O}_A(U)\) by \(s\) is a section if \(s : U \to \coprod_{P \in U} A_P\) with for all \(P\) there exists an open neighborhood \(V\) and \(a, b \in A\) such that for all \(Q \in V\) we have \(b \notin Q\) and \(s(Q) = a/b\).

The set of morphisms \((\text{Spec}A, \mathcal{O}_A) \to (\text{Spec}B, \mathcal{O}_B)\) is the same as the set of homomorphisms \(B \to A\).

### 2.4 Schemes

A locally ringed space \((X, \mathcal{O}_X)\) is a scheme if for every \(x\) there exists a \(U\) such that \((U, \mathcal{O}_X|_U)\) is isomorphic to an affine scheme.

If \(R\) is a graded ring \(\oplus_{d \geq 0} R_d\), then we define a scheme (Proj\(R, \mathcal{O}_R\)) by Proj\(R\) is the set of homogeneous prime ideals in \(R\). We want to set up \(\mathcal{O}_{R,P}\) to be \(R_{(P)}\), which is the set of elements of degree zero in \(T^{-1}R\), where \(T\) is the set of homogeneous elements which are not in \(P\). We set \(s \in \mathcal{O}_R(U)\) if \(s : U \to \coprod_{P \in U} R_{(P)}\) such that for all \(P \in U\) there exists \(V\) and \(a, b\) homogeneous of the same degree such that for all \(Q \in V\), \(b \notin Q\) and \(s(Q) = a/b\).

### 3 Lecture 3 - Projective Schemes

Let \(R\) be a graded ring, look at (Proj\(R, \mathcal{O}_R\)). Let \(f \in \oplus_{d \geq 0} R_d = R_+\), then we define \(D_+(f)\) to be the homogeneous primes not containing \(f\). \((D_+(f), \mathcal{O}_R|_{D_+(f)})\) is isomorphic to \((\text{Spec}R_{(f)}, \mathcal{O}_{R_{(f)}})\).
For any ring \( R = A[x_0, \ldots, x_n] \), we have \( (\text{Proj} R, \mathcal{O}_R) \) and we’ll call it \( \mathbb{P}^n_A \).

3.1 Gluing Schemes

Let \((X_1, \mathcal{O}_1)\) and \((X_2, \mathcal{O}_2)\) be two schemes such that \((U_1, \mathcal{O}_1|_{U_1}) \to (U_2, \mathcal{O}_2|_{U_2})\) is an isomorphism. Then we can construct a new scheme by identifying them along this map.

3.2 Schemes over a scheme

Let \(\phi : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)\). We call this an \( S \)-scheme, and often abuse notation by calling \( X \) an \( S \)-scheme.

In particular, we will look at schemes over \((\text{Spec} \, k, k)\).

3.3 Varieties and Schemes

Any variety is covered by a finite number of affine algebraic varieties.

This means that we can take any variety \( V \) over \( k \), and make a scheme over \( \text{Spec} \, k \) out of it, by just taking affine varieties to \( \text{Spec} \, A(E) \).

Now, we say that a scheme is connected if \( X \) is, irreducible if \( X \) is, it is reduced if the rings are all reduced (have no nilpotents) and integral similarly.

A scheme is integral if and only if it is reduced and irreducible.

A scheme is locally noetherian if it has a covering by spectra of noetherian rings.

Now, let \( f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) be a morphism of schemes.

We say \( f \) is locally of finite type if \((Y, \mathcal{O}_Y)\) is covered by \((\text{Spec} \, A_i, \mathcal{O}_{A_i})\) such that \( f^{-1}(\text{Spec} \, A_i) = \text{Spec} \, B_{ij} \) with the \( B_{ij} \) being \( A_i \)-algebras of finite type, that is, are finitely generated as algebras. We say that it is of finite type if the \( B_{ij} \) are finite as modules.

An example is the map from a parabola to a line, which induces \( k[x] \to k[x, y]/(y - x^2) \).

We can construct fiber products, take \( \phi : X \to S \) and \( \psi : Y \to S \), we can get \( X \times_S Y \), it’s the unique scheme such that for all maps \( Z \to X \) and \( Z \to Y \) that are equal after composing with the maps to \( S \), we get a unique map \( Z \to X \times_S Y \).

This allows us to define base change: If we have a map \( X \to Y \) and another \( Y' \to Y \), we can define \( X' = X \times_Y Y' \) and we have \( X' \to Y' \), the base change of the morphism.

For any \( X \to Y \), we have a map \( X \to X \times_Y X \) by using the same map to create the fiber product. If this morphism is closed, then we say that \( f \) is separated.

We call a morphism proper if it is separated, of finite type and universally closed, and we say that a scheme which is proper over \( \text{Spec} \, k \) is complete.
3.4 Projective Morphisms

Let \((Y, \mathcal{O}_Y)\) be a scheme. We define \(\mathbb{P}^n_Y \to Y\) to be projective space over \(Y\), where \(\mathbb{P}^n_Y = Y \times_{\text{Spec} \mathbb{Z}} \mathbb{P}^n_Z\).

We say that a morphism is projective if it factors through \(\mathbb{P}^n_Y \to Y\) and the map \(X \to \mathbb{P}^n_Y\) is a closed immersion.

Projective morphisms of Noetherian schemes are proper, and quasi-projective morphisms are separated and of finite type.

So a variety over \(k\) turns out to just be a scheme over \(k\) which is integral and of finite type.

4 Lecture 4 - Sheaves of Modules

Let \((X, \mathcal{O}_X)\) be a ringed space and \(\mathcal{M}\) a sheaf on \(X\) such that for all \(U\), \(\mathcal{M}(U)\) is an \(\mathcal{O}_X(U)\)-module and these structures are compatible with the restriction maps.

4.1 Locally Free Modules

\(\mathcal{M}\) is free if for all \(U\), \(\mathcal{M}(U)\) is a free \(\mathcal{O}_X(U)\)-module.

On a scheme \((X, \mathcal{O}_X)\), say \((\text{Spec} \mathbb{A}, \mathcal{O}_{\mathbb{A}})\), then an \(\mathcal{O}_{\mathbb{A}}\)-module on \(\text{Spec} \mathbb{A}\) is given by an \(\mathbb{A}\)-module \(M\) in the following way. Define \(\tilde{M}\) by the sections over \(U\) are \(s: U \to \prod_{P \in U} \mathcal{M}_P\) such that for all \(P \in U\), there exists \(V, m, a\) such that for all \(Q \in V\), we have \(s(Q) = \frac{m}{a}\). Then \(\tilde{M}_P = \mathcal{M}_P\). These are the prototypes of ”good” sheaves.

Definition 4.1 (Quasi-Coherent). A sheaf of \(\mathcal{O}_X\)-modules is quasi-coherent if there exists covering by open affines such that it’s restrictions are of the form \(M_i\).

Definition 4.2 (Coherent). \(\mathcal{F}\) is coherent if the \(M_i\) are finite \(\mathbb{A}_i\)-modules.

4.2 Differential Forms

Let \(A\) be a ring, \(B\) an \(A\)-algebra and \(M\) a \(B\)-module. Then an \(A\)-derivation of \(B\) into \(M\) is a map \(d: B \to M\) such that \(d\) is additive, for all \(b, b' \in B\) we have \(d(bb') = bd(b') + b'd(b)\) and for all \(a \in A\), we have \(d(a1) = 0\).

There exists a universal object, a module \(\Omega_{B/A}\) and derivation \(\delta: B \to \Omega_{B/A}\) such that any other derivation factors through it. We can construct it by looking at \(\Delta: B \otimes_A B \to B\) given by \(\Delta(b \otimes b') = bb'\), setting \(I = \ker \Delta\) and then \(\Omega_{B/A} = I/I^2\), with \(\delta: B \to I/I^2\) given by \(\delta(b) = 1 \otimes b - b \otimes 1\) modulo \(I^2\).

Now, letting \(X \to Y\) be a morphism of schemes, we can take \(\Delta: X \to X \times_Y Y\) and let \(\mathcal{F}\) be the ideal sheaf of the image of \(\Delta\). Then \(\Omega_{X/Y} = \mathcal{F}/\mathcal{F}^2\).

Let \(A \xrightarrow{h} B \xrightarrow{k} C\). Then we have exact sequences \(\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{B/A} \to 0\) and if \(I\) is an ideal of \(B\) and \(C = B/I\), then we have \(I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0\).
The sheaf is well-behaved with respect to base change: let $f : X \to Y$ a morphism, and base change along $g : Y' \to Y$, then $\Omega_{X'/Y'} = (g')^*(\Omega_{X/Y})$.

More generally, let $f : X \to Y$ and $g : Y \to Z$ morphisms of schemes. Then $f^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$ is exact, and, if $Z \subset X$ is a closed subscheme with ideal $\mathcal{I}$, we have $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \Omega_{X/Z} \to 0$.

### 4.3 Nonsingular Varieties

Let $Y \subset k^n$ affine, with $I(Y) = (f_1, \ldots, f_k)$. Then we say that $\dim Y$ is the Krull dimension of $A(Y)$, and we’ll denote it by $r$. Then a point $x$ is a nonsingular point of $Y$ if and only if $\text{rank} \left( \frac{\partial f_i}{\partial x_j} \right) = n - r$ at $x$. We say that $Y$ is nonsingular if it is nonsingular at every point.

It is a theorem that $y \in Y$ is nonsingular if and only if $\mathcal{O}_{Y,y}$ is a regular local ring, so we take this as the definition, on schemes.

What does it mean to be regular? For a Noetherian local ring of dimension $r$, the following are equivalent and taken to define regularity:

1. $\mathfrak{m}$ is generated by $r$ elements.

2. The associated graded ring with respect to $\mathfrak{m}$ is a polynomial ring in $r$ variables.

3. $\dim(\mathfrak{m}/\mathfrak{m}^2) = r$.

Now, let $X$ be a locally ringed space isomorphic to an analytic space. This is locally a local analytic space, and it is Hausdorff, and there exists an analyti-ication functor which takes varieties to analytic spaces.