1 Lecture 1

1.1 Manifolds

Definition 1.1 (Locally Euclidean). A topological space is locally Euclidean if every \( p \in M \) has a neighborhood \( U \) and a homeomorphism \( \phi : U \rightarrow V \), where \( V \) is an open subset of \( \mathbb{R}^n \). We call the pair \((U, \phi)\) a chart.

Definition 1.2 (\( C^\infty \) Compatible). Two charts are \( C^\infty \) compatible if \( \phi \circ \psi^{-1} \) and \( \psi \circ \phi^{-1} \) are \( C^\infty \) functions.

Definition 1.3 (Manifold). A manifold is a locally Euclidean, Hausdorff, second countable topological space on which there exists a covering by \( C^\infty \) compatible charts.

Example 1.4 (The Circle). The circle is given by \( x^2 + y^2 = 1 \). Taking the upper and lower semicircles, along with the left and right semicircles, provides the desired cover.

1.2 Tangent Space

Let \( r^i \) be coordinates on an open set in \( \mathbb{R}^n \) and \( x^i \) be \( r^i \circ \phi \).

Definition 1.5 (\( C^\infty \) function). \( f : M \rightarrow \mathbb{R} \) is \( C^\infty \) at \( p \) if there exists \( (U, \phi) \) with \( p \in U \) such that \( f \circ \phi^{-1} \) is \( C^\infty \) at \( \phi(p) \in \mathbb{R}^n \).

Definition 1.6 (Partial Derivatives). If \( f \in C^\infty(M) \) and \( (U, x^1, \ldots, x^n) \) is a chart containing \( p \), then we define \( \frac{\partial}{\partial x^i}|_p f = \frac{\partial}{\partial r^i}|_{\phi(p)} f \circ \phi^{-1} \).

Definition 1.7 (Tangent Space). \( T_p M \), the tangent space at \( p \), is the vector space with basis \( \frac{\partial}{\partial x^i} \) at \( p \in (U, \phi) \).

Theorem 1.8. Let \( x^i \) and \( y^j \) be different sets of coordinates at a point \( p \). Then
\[
\frac{\partial}{\partial y^j} = \sum a^i_j \frac{\partial}{\partial x^i}, \quad \text{where} \quad a^i_j = \frac{\partial x^i}{\partial y^j}.
\]
The proof is a direct computation.

Definition 1.9 (Vector Field). A vector field is a function \( X : M \rightarrow \bigoplus_{p\in M} T_p M \) such that \( X_p \in T_p M \). We can write it at \( p \) as \( X = \sum a^i \frac{\partial}{\partial x^i} \) and we call \( X \) a \( C^\infty \) vector field if the \( a^i \) are all \( C^\infty \).
1.3 Differential Forms

Definition 1.10 (Cotangent Space). The cotangent space is $T^*_pM = \text{hom}(T_pM, \mathbb{R})$. It has basis $(dx^i)_p$ the duals of $\frac{\partial}{\partial x^i}$’s at $p$.

Definition 1.11 (Differential 1-Form). A 1-form is a function $\omega : M \to \bigoplus_{p \in M} T^*_pM$ such that $\omega_p \in T^*_pM$. We can write them near $p$ as $\omega = \sum a^i(dx^i)_p$ and call them $C^\infty$ if the $a^i$ are.

Definition 1.12 (Wedge Product). Let $V$ be a real vector space, and let $\alpha, \beta \in V^*$, then $(\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v)$.

It then follows that $\beta \wedge \alpha = -\alpha \wedge \beta$ and $\alpha \wedge \alpha = 0$.

Definition 1.13 (Wedge Product). Let $A_k(V)$ be the set of alternating functions on $V$ with $k$ inputs. If $\alpha \in A_k$ and $\beta \in A_\ell$ then $\alpha \wedge \beta \in A_{k+\ell}$ and $\beta \wedge \alpha = (-1)^{k\ell}\alpha \wedge \beta$.

We define $A_0(V) = \mathbb{R}$.

Definition 1.14 $(k$-form). A $k$-form on $M$ is a map $\omega : M \to \bigoplus_{p \in M} A_k(T^*_pM)$ such that $\omega_p \in A_k(T^*_pM)$. It is called $C^\infty$ if the coefficient functions are.

We denote $\mathcal{A}^i(M)$ to be the $C^\infty$ $i$-forms on $M$. We have that $\mathcal{A}^0(M) = C^\infty(M)$, and $\mathcal{A}^k(M) = 0$ for $k$ greater than the dimension of $M$.

1.4 Exterior Derivative

Definition 1.15 (Derivative). If $f \in C^\infty(M)$, define a 1-form $df \in \mathcal{A}^1(M)$ on a chart $(U, x^i)$ by $df = \sum \frac{\partial f}{\partial x^i} dx^i$. More generally, let $\omega \in \mathcal{A}^k(M)$ by $\omega = \sum a_I dx^I$, then we define $d\omega = \sum d(a_I) \wedge dx^I$.

Proposition 1.16. The exterior derivative $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ satisfies

1. Antiderivation: $d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \tau} \omega \wedge d\tau$
2. $d^2 = 0$.

1.5 DeRham Cohomology

We define the kernel of $d$ to be the closed $k$-forms $Z^k(M)$ and the image to be the exact $k$-forms, $B^k(M)$.

Definition 1.17 (DeRham Cohomology). $H^k(M) = Z^k(M)/B^k(M)$.

Example 1.18 ($H^0(M)$). $H^0(M)$ is just $\ker d : C^\infty(M) \to \mathcal{A}^1(M)$, because there is no $\mathcal{A}^{-1}(M)$. So we end up with the set of locally constant functions on $M$, and, because $M$ is a manifold, $H^0(M)$ is the set of all functions constant on the connected components.
Example 1.20 \((H^k(\mathbb{R}))\). An element of \(A^1(\mathbb{R})\) is just \(f(x)dx\) where \(f \in C^\infty(\mathbb{R})\), and element of \(dA^0(\mathbb{R})\) is \(dg = g'(x)dx\). So is every \(C^\infty\) function equal to \(g'(x)\) for some \(g \in C^\infty(\mathbb{R})\)? Yes, by the fundamental theorem of calculus, so \(H^k(\mathbb{R})\) is \(\mathbb{R}\) for \(k = 0\) and is 0 else.

2 Lecture 2 - Computation of deRham Cohomology

2.1 Pullback of Forms

If \(f : M \to N\) is a \(C^\infty\) map, then there is a pullback map \(f^* : \mathcal{A}^k(N) \to \mathcal{A}^k(M)\).

For \(k = 0\), then \(f^*(h) = h \circ k\). In generally, locally \(\omega \in \mathcal{A}^k(N)\) can be written \(\omega = \sum a_idx^j = \sum a_idx^{j_1} \wedge \ldots \wedge dx^{j_k}\) and we define \(f^*\omega = \sum (f^*a_i)(d(f^*x^{j_1}) \wedge \ldots \wedge d(f^*x^{j_k})\).

Proposition 2.1.

1. \(1^*_M = 1_{\mathcal{A}^k(M)}\)
2. \((f \circ g)^* = g^* \circ f^*\)
3. \(f^*d = df^*\)

By 3, we know that \(f^*\) of a closed form is closed. If \(d\omega = 0\), then \(d(f^*\omega) = f^*d\omega = 0\). Now, \(f^*(d\tau) = d(f^*\tau)\), and so exact forms pullback to exact forms. Thus, \(f^*\) induces a map (also denoted by \(f^*\) on cohomology \(H^k(N) \to H^k(M)\)).

If \(f : M \to N\) is a diffeomorphism, then there exists \(g : N \to M\) such that \(f \circ g = 1_N\) and \(g \circ f = 1_M\), then \(g^* \circ f^* = 1_{H^*(N)}\) and \(f^* \circ g^* = 1_{H^*(M)}\).

Theorem 2.2. If \(f : M \to N\) is a diffeomorphism, then \(f^* : H^*(N) \to H^*(M)\) is an isomorphism.

2.2 Homological Algebra

A cochain complex \(C\) is \(0 \to C^0 \xrightarrow{d} C^1 \to \ldots\) such that \(d^2 = 0\). So \(H^k(C) = \ker d_k/\text{im}d_{k-1}\) is defined.

A sequence of vector spaces \(A \xrightarrow{i} B \xrightarrow{j} C\) is exact at \(B\) if \(\text{im}(i) = \ker j\).

A linear map \(f : (A,d) \to (B,d)\) of cochain complexes is a cochain map if \(f \circ d = d \circ f\) (this is the data of a map in each degree).

A cochain map induces \(f^* : H^k(A) \to H^k(B)\).

Theorem 2.3 (Zig-Zag Lemma). A short exact sequence of cochain complexes \(0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0\) induces a long exact sequence on cohomology \(\ldots \to H^k(A) \to H^k(B) \to H^k(C) \xrightarrow{d^*} H^{k+1}(A) \to \ldots\).
2.3 Mayer-Vietoris Sequence

Suppose that $M$ is covered by open sets $U, V$ so that $M = U \cup V$. Then we have $\mathcal{A}^k(M) \xrightarrow{\partial} \mathcal{A}^k(U) \oplus \mathcal{A}^k(V) \xrightarrow{\partial} \mathcal{A}^k(U \cap V)$ by the restriction maps followed by the difference of restrictions.

**Theorem 2.4** (Mayer-Vietoris Sequence). The sequence above is exact.

2.4 $H^*(S^1)$

We cover $S^1$ by the union of two copies of $\mathbb{R}$. Then $H^*(U) = H^*(V) = H^*(\mathbb{R})$ and $H^*(U \cap V)$ is two copies of $\mathbb{R}$, and so $H^*(U \cap V) = H^*(\mathbb{R}) \oplus H^*(\mathbb{R})$.

Then the Mayer-Vietoris sequence becomes $0 \rightarrow \mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow H^1(S^1) \rightarrow 0 \rightarrow 0 \rightarrow \ldots$ So by exactness, $H^1(S^1) = \text{im}(d^*) \cong \mathbb{R}^2 / \ker d^* = \mathbb{R}^2 / \text{im}(j^*) \cong \mathbb{R}$, because the image of $j^*$ is the diagonal.

2.5 Smooth Homotopy

Two $C^\infty$ maps $f_0, f_1 : M \to N$ are smoothly homotopic if there exist a $C^\infty$ map $F : M \times [0, 1] \to N$ such that $F(x, 0) = f_0(x)$ and $f(x, 1) = f_1(x)$ (where we say that $F$ is $C^\infty$ if it can be extended to a $C^\infty$ function in a neighborhood of $M \times [0, 1]$ in $M \times \mathbb{R}$). We write $f_0 \sim f_1$.

**Definition 2.5** (Homotopy Inverse). $f : M \to N$ has a homotopy inverse if there exists $g : N \to M$ such that $f \circ g \sim 1_N$ and $g \circ f \sim 1_M$.

Then we say that $M$ and $N$ have the same homotopy type and $f$ is a homotopy equivalence.

**Homotopy Axiom.** Homotopic maps $f, g : M \to N$ induce the same map on cohomology $f^* = g^* : H^*(N) \to H^*(M)$.

**Example 2.6.** $\mathbb{R}^n$ has the homotopy type of a point $\{0\}$. We have $i$ the inclusion of the origin and $\pi$ the unique map to the point. Then $\pi \circ i$ is the identity map on the point, and $i \circ \pi : \mathbb{R}^n \to \mathbb{R}^n$ sends every point to 0. We claim this is homotopic to the identity. Define $F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ by $F(x, t) = (1 - t)x$ to be the homotopy.

**Corollary 2.7.** If $f : M \to N$ is a homotopy equivalence, then $f^* : H^*(N) \to H^*(M)$ is an isomorphism.

**Corollary 2.8** (Poincare Lemma). $H^*(\mathbb{R}^n) = H^*(\text{pt})$ is $\mathbb{R}$ in degree 0 and 0 else.

3 Lecture 3 - Presheaves and Cech Cohomology

**Definition 3.1** (Presheaf). A presheaf on a topological space $X$ is a function that assigns to each open $U \subset X$ an abelian group $\mathcal{F}(U)$ and to every inclusion $i_U^V : U \to V$ a group homomorphism $\mathcal{F}(i_U^V) = \rho_U^V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho_U^U = 1_{\mathcal{F}(U)}$, and $\rho_W^V \circ \rho_U^W = \rho_U^V$.
Example 3.2. \( \mathcal{A}^k(U) \) is the \( C^\infty \) \( k \)-forms on \( U \). This is a presheaf on a manifold \( M \).

Example 3.3. If \( G \) is an abelian group, for every open \( U \subset X \), define \( G(U) \) to be the locally constant functions \( f : U \to G \). Then \( G \) is a presheaf.

3.1 Čech Cohomology of an Open Cover

Let \( \{ U_\alpha \} \) be an open cover of a topological space indexed by a totally ordered set. We’ll denote intersections by putting the subscripts together.

When \( \alpha = 0, 1 \) gives the cover, we have Mayer-Vietoris, which says \( 0 \to \mathcal{A}^k(M) \to \prod \mathcal{A}^k(U_\alpha) \to \mathcal{A}^k(U_{01}) \to 0 \).

Now, let \( F \) be a presheaf on a topological space \( X \). We then have a sequence

\[ 0 \to F(X) \to \prod_\alpha F(U_\alpha) \to \prod_{\alpha < \beta} F(U_{\alpha \beta}) \to \ldots \]

Define \( C^p(U, \mathcal{F}) \) to be the term involving \( p \) open sets. Then we define \( \delta : C^p(U, \mathcal{F}) \to C^{p+1}(U, \mathcal{F}) \) by \( (\delta \omega)_{a_0, \ldots, a_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{a_0, \ldots, \hat{a}_i, \ldots, a_{p+1}} \).

It turns out that \( \delta^2 = 0 \), and so we define the cohomology of this complex to be the Čech cohomology \( H^k(U, \mathcal{F}) \).

3.2 DirectLimits

Definition 3.4 (Directed Set). A directed set is a set \( I \) with a binary relation \(<\) that is reflexive, transitive and such that any two elements have a common upper bound.

Example 3.5. Fix \( p \in X \). Let \( I \) be the set of neighborhoods of \( p \) in \( X \) and say that \( U < V \) iff \( U \subset V \).

Fix a topological space \( X \). Then \( I \) be the set of all open covers of \( X \). An open cover \( V \) refines \( U \) if every \( V \in V \) is contained in some \( U \in U \). Refinement gives a directed set structure to the set of covers. A refinement \( V \) of \( U \) can be given by a refinement map on the index sets stating which \( U \) each \( V \) is contained in.

A directed system of groups is a collection \( \{ G_i \}_{i \in I} \) of groups indexed by a directed set \( U \) such that for all \( a < b \) we have a homomorphism \( f_{ab} : G_a \to G_b \) satisfying that \( f_{aa} = 1 \) and \( f_{ac} = f_{bc} \circ f_{ab} \).

Example 3.6. Let \( I \) be the neighborhoods of \( p \in X \). Then set \( G_U = C^\infty(U) \) and say that \((U, f)\) and \((V, g)\) are equivalent iff there exists \( W \subset U \cap V \) such that \( f|_W = g|_W \). We call these the germs of functions at \( p \).

In \( \coprod_{i \in I} G_i \), let \( g_a \in G_a \) and \( g_b \in G_b \). Then we say \( g_a \sim g_b \) if there exists \( c > a, b \) such that \( f_{ac}(g_a) = f_{bc}(g_b) \) and define \( \lim_{i \in I} G_i = \coprod G_i / \sim \).
3.3 Čech Cohomology of a Topological Space

For each open cover, we have $\hat{H}^k(U, \mathcal{F})$, and we have restrictions making it into a directed system of abelian groups. So we define $\hat{H}^k(X, \mathcal{F})$ to be the limit of this system.

3.4 $C^\infty$ partitions of unity

Definition 3.7 (Partition of Unity). A $C^\infty$ partition of unity on a manifold $M$ is a collection of $C^\infty$ functions $\rho_\alpha$ with $\sum \rho_\alpha = 1$ and $\rho_\alpha : M \to [0,1]$.

We define the support of a function to be the set where it is nonzero.

A collection of subsets $\{A_\alpha\}$ in $X$ is locally finite if every $p \in X$ has a neighborhood that meets only finitely many of the $A_\alpha$.

Theorem 3.8. Given any open cover of a manifold, there exists a $C^\infty$ partition of unity with each element’s support contained in one of the open sets of the cover.

4 Lecture 4 - Sheaves and the Čech-deRham Isomorphism

Version 6 of lecture notes on Summer School website.

Exercise 4.1. Compute $H^*$ of $S^n$, $\mathbb{R}^n \setminus \{0\}$, $\mathbb{C}P^1$, $\mathbb{C}P^2$ for problem session today.

Problem Session: Does $H^*(U, \mathcal{F})$ depend on the order on the index set?

4.1 Sheaves

Definition 4.2 (Sheaf). A sheaf on a topological space $X$ is a presheaf such that for any open set $U$ and any open cover $U_i$ of $U$, we have

1. Uniqueness: If $s \in \mathcal{F}(U)$ such that $s|_{U_i} = 0$ for all $i$, then $s = 0$.

2. Gluing: If $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j$, then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

Example 4.3.

1. $A^k$ is a sheaf for any manifold.

2. $Z^k$, the closed $C^\infty$ $k$-forms on a manifold is a sheaf.

3. $A^{p,d}$ is a sheaf for any complex manifold.

4. $\mathbb{R}$, the locally constant functions to $\mathbb{R}$, is a sheaf.

However, the presheaf of constant functions is not a sheaf.
4.2 Cohomology in Degree 0

Let $F$ be a sheaf on a topological space $X$. We want to know $\check{H}^0(U, F)$. Let $U = \{U_i\}$ be an open cover of $X$. Then as $F$ is a sheaf, we have the Cech sequence.

So then $\check{H}^0(U, F) = \ker \delta$ as $C^{-1} = 0$ which is just $\text{im}(r)$ where $r$ is the restriction map $F(X) \to \prod F(U_i)$. But this is precisely $F(X)$.

Now, let $V$ be a refinement of $U$. Both Cech groups are isomorphic to $F(X)$, and so they are isomorphic.

**Theorem 4.4.** If $F$ is a sheaf on a topological space $X$, then $\check{H}^0(X, F) = F(X)$.

4.3 Further Computations

For all $q$ and all $k > 0$, $\check{H}^k(M, A^q) = 0$. But how do we prove this? The standard method for proving that cohomology is zero is finding a map $K : C^p \to C^{p-1}$ such that $1 - 0 = \delta K + K \delta$. Applying $(\delta K + K \delta)$ to a cocycle just gives $\delta K$. Therefore, this induces the zero map on cohomology, and so $1^* = 0$, which can only happen when $H^k = 0$. We call $K$ a chain homotopy. There is a theorem guaranteeing that such a $K$ exists when the cohomology is zero.

Let $U = \{U_\alpha\}$ be an open cover of $M$, and $\{\rho_\alpha\}$ be a $C^\infty$ partition of unity subordinate to $U$. For $k \geq 1$, then $K : C^p(U, F) \to C^{p-1}(U, F)$ by $(K \omega)_{\alpha_0, \ldots, \alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha_0, \alpha_1, \ldots, \alpha_{p-1}}$.

We must check that $\delta K + K \delta = 1$ for $k \geq 1$, and then we have the vanishing of the Cech cohomology, when we take $\mathcal{F} = A^0$. We in fact have $H^k(M, A^{p,q}) = 0$ for $k \geq 1$ on a complex manifold.

But $H^k(M, \Omega^q)$ is not necessarily zero!

4.4 Sheaf Morphism

A morphism of sheaves is a collection of maps, one for each open set, compatible with the restriction maps.

We define the stalk of a presheaf $\mathcal{F}_p$ at $p \in X$ to be $\lim_{U \ni p} \mathcal{F}(U)$. So a presheaf homomorphism $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ by $(U, s) \mapsto (U, \phi(s))$.

4.5 Exact Sequences of Sheaves

**Theorem 4.5.** A short exact sequence of sheaves $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ induces a long exact sequence $\ldots \to H^k(X, \mathcal{E}) \to H^k(X, \mathcal{F}) \to H^k(X, \mathcal{G}) \to H^{k+1}(X, \mathcal{E}) \to \ldots$.

**Theorem 4.6 (Poincaré Lemma).** The sequence of sheaves $0 \to \mathbb{R} \to A^0 \to \ldots$ is exact.

**Proof.** Look at $0 \to \mathbb{R} \to A^0 \to \text{im}(d_0) \to 0$, and similarly, we can make $0 \to Z^k \to A^k \to Z^{k-1} \to 0$ for all $k$. 


So we have $0 \to H^{k-1}(M, Z^1) \to H^k(M, \mathbb{R}) \to H^k(M, A^0) = 0$, and we can finish the proof by basic homological manipulations. \qed