1 Affine Varieties

We will begin following Kempf's Algebraic Varieties, and eventually will do things more like in Hartshorne. We will also use various sources for commutative algebra.

What is algebraic geometry? Classically, it is the study of the zero sets of polynomials.

We will now fix some notation. k will be some fixed algebraically closed field, any ring is commutative with identity, ring homomorphisms preserve identity, and a k-algebra is a ring R which contains k (i.e., we have a ring homomorphism $\iota : k \to R$).

$P \subseteq R$ an ideal is prime iff R/P is an integral domain.

Algebraic Sets

We define affine n-space, $\mathbb{A}^n = k^n = \{(a_1, \ldots, a_n) : a_i \in k\}$.

Any $f = f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ defines a function $f : \mathbb{A}^n \to k : (a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n)$.

Exercise If $f, g \in k[x_1, \ldots, x_n]$ define the same function then $f = g$ as polynomials.

Definition 1.1 (Algebraic Sets). *Let $S \subseteq k[x_1, \ldots, x_n]$ be any subset. Then $V(S) = \{a \in \mathbb{A}^n : f(a) = 0$ for all $f \in S\}$.*

A subset of $\mathbb{A}^n$ is called algebraic if it is of this form.

e.g., a point $\{(a_1, \ldots, a_n)\} = V(x_1 - a_1, \ldots, x_n - a_n)$.

Exercises

1. $I = (S)$ is the ideal generated by S. Then $V(S) = V(I)$.
2. $I \subseteq J \Rightarrow V(J) \subseteq V(I)$.
3. $V(\cup_\alpha I_\alpha) = V(\sum I_\alpha) = \cap V(I_\alpha)$.
4. $V(I \cap J) = V(I \cdot J) = V(I) \cup V(J)$.

Definition 1.2 (Zariski Topology). *We can define a topology on $\mathbb{A}^n$ by defining the closed subsets to be the algebraic subsets. $U \subseteq \mathbb{A}^n$ is open iff $\mathbb{A}^n \setminus U = V(S)$ for some $S \subseteq k[x_1, \ldots, x_n]$.*

Exercises 3 and 4 imply that this is a topology.

The closed subsets of $\mathbb{A}^1$ are the finite subsets and $\mathbb{A}^1$ itself.

Definition 1.3 (Ideal of a Subset). *If $W \subset \mathbb{A}^n$ is any subset, then $I(W) = \{f \in k[x_1, \ldots, x_n] : f(a) = 0$ for all $a \in W\}$*

Facts/Exercises

1. $V \subseteq W \Rightarrow I(W) \subseteq I(V)$
2. $I(\emptyset) = (1) = k[x_1, \ldots, x_n]$
3. $I(\mathbb{A}^n) = (0)$.

Definition 1.4 (Affine Coordinate Ring)**.** *$W \subset \mathbb{A}^n$ is algebraic. Then $A(W) = k[W] = k[x_1, \ldots, x_n]/I(W)$*

We can think of this as the ring of all polynomial functions $f : W \to k$.

Definition 1.5 (Radical Ideal)**.** *Let R be a ring and $I \subseteq R$ be an ideal, then the radical of I is the ideal $\sqrt{I} = \{f \in R : f^i \in I$ for some $i \in \mathbb{N}\}$*

We call I a radical ideal if $I = \sqrt{I}$.

Exercise

If I is an ideal, then $\sqrt{I}$ is a radical ideal.

Proposition 1.1. *$W \subseteq \mathbb{A}^n$ any subset, then $I(W)$ is a radical ideal.*

Proof. We have that $I(W) \subseteq \sqrt{I(W)}$.

Suppose $f \in \sqrt{I(W)}$. Then $f^i \in I$ for some i. That is, for all $a \in W$, $f^i(a) = 0$. Thus, $f(a)^m = 0 = f(a)$. And so, $f(a) \in I$. □

Exercises

1. $S \subseteq k[x_1, \ldots, x_n]$, then $S \subseteq I(V(S))$.
2. $W \subseteq \mathbb{A}^n$ then $W \subseteq V(I(W))$.
3. $W \subseteq \mathbb{A}^n$ is an algebraic subset, then $W = V(I(W))$.
4. $I \subseteq k[x_1, \ldots, x_n]$ is any ideal, then $V(I) = V(\sqrt{I})$ and $\sqrt{I} \subseteq I(V(I))$

Theorem 1.2 (Nullstellensatz)**.** *Let k be an algebraically closed field, and $I \subseteq k[x_1, \ldots, x_n]$ is an ideal, then $\sqrt{I} = I(V(I))$.*

Corollary 1.3. *$k[V(I)] = k[x_1, \ldots, x_n]/\sqrt{I}$.*

To prove the Nullstellensatz, we will need the following:

Theorem 1.4 (Nöther's Normalization Theorem)**.** *If R is any finitely generated k-algebra (k can be any field), then there exist $y_1, \ldots, y_m \in R$ such that $y_1, \ldots, y_m$ are algebraically independent over k and R is an integral extension of the subring $k[y_1, \ldots, y_m]$.*

Proof is in Eisenbud and other Commutative Algebra texts.

Theorem 1.5 (Weak Nullstellensatz)**.** *Let k be an algebraically closed field, and $I \subsetneq k[x_1, \ldots, x_n]$ any proper ideal, then $V(I) \neq \emptyset$.*

Proof. We may assume without loss of generality that I is actually a maximal ideal. Then $R = k[x_1, \ldots, x_n]/I$ is a field. R is also a finitely generated k-algebra, and so by Normalization, $\exists y_1, \ldots, y_m \in R$ such that $y_1, \ldots, y_m$ are algebraically independent over k and that R is integral over $k[y_1, \ldots, y_m]$.

Claim: $m = 0$. Otherwise, $y_1^{-1} \in R$ is integral over $k[y_1, \ldots, y_m]$, and so then $y_1^{-p} + y_1^{1-p} f_1 + \ldots + y_1^{-1} f_{p-1} + f_p = 0$ for $f_i \in k[y_1, \ldots, y_m]$. Multiplying through by y_1^p gives $1 = -(y_1 f_1 + \ldots + y_1^{p-1} f_{p-1} + y_1^p f_p) \in (y_1)$, which contradicts the algebraic independence.

Thus, the field R is algebraic over k. As k is algebraically closed, $R = k$. $k \subseteq k[x_1, \ldots, x_n] \to R = k$

Let $a_i =$ the image in k of x_i. Then $x_i - a_i \in I$. Thus, the ideal generated by $(x_1 - a_1, \ldots, x_n - a_n) \subseteq I \subsetneq k[x_1, \ldots, x_n]$, and so they $I = (x_1 - a_1, \ldots, x_n - a_n)$, as it is a maximal ideal.

$V(I) = V(x_1 - a_1, \ldots, x_n - a_n) = \{(a_1, \ldots, a_n)\} \neq \emptyset$ □

Note: Any maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ with $a_i \in k$.

This is NOT true over $\mathbb{R}$, look at the ideal $(x^2 + 1) \subseteq \mathbb{R}[x]$. It is, in fact, maximal.

Now, we can prove the Nullstellensatz.

Proof. Let $I \subseteq k[x_1, \ldots, x_n]$ be any ideal. We will prove that $I(V(I)) = \sqrt{I}$.

It was an exercise that $\sqrt{I} \subseteq I(V(I))$.

Let $f \in I(V(I))$. We must show that $f \in \sqrt{I}$.

Looking at $\mathbb{A}^{n+1}$, we have the variables, $x_1, \ldots, x_n, y$. Set $J = (I, 1 - yf) \subseteq k[x_1, \ldots, x_n, y]$.

Claim: $V(J) = \emptyset \subset \mathbb{A}^{n+1}$. This is as, if $p = (a_1, \ldots, a_n, p) \in V(J)$, then $(a_1, \ldots, a_n) \in V(I)$, then $(1 - yf)(p) = 1 - bf(a_1, \ldots, a_n)$. But $f(a_1, \ldots, a_n) = 0$, so $(1 - yf)(p) = 1$, and so $p \notin V(J)$.

By the Weak Nullstellensatz, $J = k[x_1, \ldots, x_n, y]$. Thus $1 = h_1 g_1 + \ldots + h_m g_m + q(1 - yf)$ where $g_1, \ldots, g_m \in I$ and $h_1, \ldots, h_m, q \in k[x_1, \ldots, x_n, y]$.

Set $y = f^{-1}$, and multiply by some big power of f to get a polynomial equation once more.

Then $f^N = \tilde{h}_1 g_1 + \ldots + \tilde{h}_m g_m$ where the $\tilde{h}_i = f^N h_i(x_1, \ldots, x_n)$.

And so, we have $f^N \in I$, and thus, $f \in \sqrt{I}$, by definition. □

exercise: $V(I(W)) = \overline{W}$ in the Zariski Topology.

Irreducible Algebraic Sets

Recall: $V(y^2 - xy - x^2 y + x^3) = V(y - x) \cup V(y - x^2)$, and $V(xz, yz) = V(x, y) \cup V(z)$.

Definition 1.6 (Reducible Subsets)**.** *A Zariski Closed subset $W \subseteq \mathbb{A}^n$ is called reducible if $W = W_1 \cup W_2$ where $W_i \subsetneq W$ and W_i closed.*

Otherwise, we say that W is irreducible.

Proposition 1.6. *Let $W \subseteq \mathbb{A}^n$ be closed. Then W is irreducible iff $I(W)$ is a prime ideal.*

Proof. $\Rightarrow$: Suppose $I(W)$ is not prime. Then $\exists f_1, f_2 \notin I(W)$ such that $f_1 f_2 \in W$. Set $W_1 = W \cap V(f_1)$ and $W_2 = W \cap V(f_2)$.

As $f_i \notin I(W)$, $W \not\subseteq V(f_i)$, and so $W_i \subsetneq W$. Now we must show that $W = W_1 \cup W_2$. Let $a \in W$. Assume $a \notin W_1$. Then $f_1(a) \neq 0$, but $f_1(a) f_2(a) = 0$, so $f_2(a) = 0$, thus $a \in W_2$.

$\Leftarrow$: Exercise □

This gives us the beginning of an algebra-geometry dictionary.

Algebra	Geometry
$k[x_1, \ldots, x_n]$	$\mathbb{A}^n$
radical ideals	closed subsets
prime ideals	irreducible closed subsets
maximal ideals	points

In fact, this is an order reversing correspondence. So $I \subseteq J \iff V(I) \supseteq V(J)$, but this requires I, J to be radical.

Definition 1.7 (Nötherian Ring)**.** *A ring R is called Nötherian if every ideal $I \subseteq R$ is finitely generated.*

Exercise A ring R is Nötherian iff every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \ldots$ stabilizes, that is, $\exists N$ such that $I_N = I_{N+1} = \ldots$.

Theorem 1.7 (Hilbert's Basis Theorem)**.** *If R is Nötherian, then $R[x]$ is Nötherian.*

Corollary 1.8. *$k[x_1, \ldots, x_n]$ is Nötherian.*

Definition 1.8 (Nötherian Topological Space)**.** *A topological space X is Nötherian if every descending chain of closed subsets stabilizes.*

Corollary 1.9. *$\mathbb{A}^n$ is Nötherian.*

$W_1 \supseteq W_2 \supseteq \ldots$ closed in $\mathbb{A}^n$, then $I(W_1) \subseteq I(W_2) \subseteq \ldots$ ideals in $k[x_1, \ldots, x_n]$, and so must stabilize.

Theorem 1.10. *Any closed subset of a Nötherian Topological Space X is a union of finitely many irreducible closed subsets.*

Proof. Assume the result is false. $\exists W$ a closed subset of X which is not the union of finitely many irreducible closed sets.

As X is Nötherian, we may assume that W is a minimal counterexample. W is not irreducible, and so $W = W_1 \cup W_2$, where $W_i \subsetneq W$ and W_i closed. The W_i can't be counterexamples, as W is a minimal one, but then $W = W_1 \cup W_2$ and each W_i is the union of finitely many irreducible closed sets. Thus, W cannot be a counterexample. □

Corollary 1.11. *Every closed $W \subseteq \mathbb{A}^n$ is union of finitely many irreducible closed subsets.*

Example: $V(xy) = V(x) \cup V(y) \cup V(x-1, y)$.

Recall: X is a topological space, then if $Y \subseteq X$ is any subset, it has the subspace topology, that is, $U \subseteq Y$ is open iff $\exists U' \subseteq X$ open such that $U = U' \cap Y$.

Note:

1. $W \subseteq Y$ is closed iff $W = \overline{W} \cap Y$, where the closure is in X.
2. X is Nötherian implies that Y is Nötherian in the subspace topology.

Definition 1.9 (Zariski Topology on $X \subseteq \mathbb{A}^n$)**.** *If $X \subseteq \mathbb{A}^n$, then the Zariski Topology on X is the subspace topology.*

Definition 1.10 (Components of X)**.** *If X is any Nötherian Topological Space, then the maximal irreducible closed subsets of X are called the (irreducible) components of X.*

Exercises

1. X has finitely many components.
2. $X =$ the union of its irreducible components.
3. $X \neq$ union of any proper subset of its components.
4. A topological space is Nötherian if and only if every subset is quasi-compact.
5. A Nötherian Hausdorff space is finite.

Recall: $X \subseteq \mathbb{A}^n$ closed. Then $A(X) = k[x_1, \ldots, x_n]/I(X)$.

Definition 1.11. *If $f \in A(X)$, set $D(f) = \{a \in X : f(a) \neq 0\}$.*

Proposition 1.12. *The sets $D(f)$ form a basis for the Zariski Topology on X.*

Proof. Let $p \in U \subseteq X$, U open. Show that $p \in D(f) \subseteq U$ for some $f \in A(X)$. $Z = X \setminus U$ a closed subset of X, and $Z \subsetneq Z \cup \{p\}$ implies that $I(Z) \supsetneq I(Z \cup \{p\})$.

Take any $f \in I(Z) \setminus I(Z \cup \{p\})$. Then f vanishes on Z but not at p, so $p \in D(f)$. □

Regular Functions

Let $X \subseteq \mathbb{A}^n$ be an algebraic subset, and $U \subseteq X$ is a relatively open subset of X.

Definition 1.12 (Regular Function)**.** *A function $f : U \to k$ is called regular if f is locally rational. That is, $\exists$ open cover $U = \cup_\alpha U_\alpha$ and functions $p_\alpha, q_\alpha \in A(X)$ such that $\forall a \in U_\alpha$, $q_\alpha(a) \neq 0$ and $f(a) = p_\alpha(a)/q_\alpha(a)$.*

We define $k[U]$ to be the set of regular functions from U to k.

Note:

1. $k[U]$ is a k-algebra.
2. $A(X) \subseteq k[X]$.

Example

Let $X = V(xy - zw) \subseteq \mathbb{A}^4$. $f : U \to k$ can be defined by $f = x/w$ on $D(w)$ and $f = z/y$ on $D(y)$. Thus, $f \in k[U]$.

Exercise; $\nexists p, q \in A(X)$ such that $q(a) \neq 0$ and $f(a) = p(a)/q(a)$ for all $a \in U$.

Lemma 1.13. *Let $q_1, \ldots, q_n \in A(X)$. Then $D(q_1) \cup \ldots \cup D(q_n) = X$ iff $(q_1, \ldots, q_m) = (1) = A(X)$.*

Proof. $\Leftarrow$: $1 = \sum h_i q_i$, $h_i \in A(X)$, then the q_i cannot all vanish at any point, and so we are done.

$\Rightarrow$: Take $Q_i \in k[x_1, \ldots, x_n]$ such that $q_i = \overline{Q_i} \in A(X)$. $D(q_1) \cup \ldots \cup D(q_m) = X$, so $X \cap V(Q_1) \cap \ldots \cap V(Q_m) = \emptyset = V(I(X), Q_1, \ldots, Q_m) = \emptyset$, and so, by the weak nullstellensatz, $(I(X), Q_1, \ldots, Q_m) = (1) \subseteq k[x_1, \ldots, x_n]$, and so $(q_1, \ldots, q_m) = (1) = A(X)$. $\square$

Theorem 1.14. *Let $X \subseteq \mathbb{A}^n$ be an algebraic set. Then $k[X] = A(X)$.*

Proof. Let $f \in k[X]$. Then $X = U_1 \cup \ldots \cup U_m$ and there are $p_i, q_i \in A(X)$ such that $q_i \neq 0$ and $f = p_i/q_i$ on U_i.

We can refine the open cover such that each $U_i = D(g_i)$ for some g_i. Note: $f = p_i/q_i = \frac{p_i g_i}{q_i g_i}$ on $U_i = D(g_i) = D(g_i q_i)$. We can replace p_i with $p_i g_i$ and q_i by $q_i g_i$.

Then we can assume that U_i is $D(q_i) = D(q_i^2)$. Thus, $X = D(q_1^2) \cup \ldots \cup D(q_m^2)$. By the lemma, we know that $1 = \sum_{i=1}^m h_i q_i^2$, $h_i \in A(X)$. Note $q_i^2 f = q_i p_i$ on U_i, and $q_i = 0$ outside of U_i.

$f = 1f = \sum_{i=1}^m h_i q_i^2 f = \sum_{i=1}^m h_i q_i p_i$, so $f \in A(X)$. $\square$

Definition 1.13 (Spaces With Functions)**.** *A space with functions (SWF) is a topological space X together with an assignment to each open $U \subseteq X$ of a $k-$algebra $k[U]$ consisting of functions $U \to k$. These are called regular functions. It must also satisfy the following:*

1. *If $U = \cup U_\alpha$ is an open cover and $f : U \to k$ any function, then f is regular on U iff $f|_{U_\alpha}$ is regular on U_α for all α.*
2. *If $U \subseteq X$ is open, $f \in k[U]$, then $D(f) = \{a \in U : f(a) \neq 0\}$ is open and $\frac{1}{f} \in k[D(f)]$.*

Note: $\mathscr{O}_X(U) = k[U]$ is another common notation.

Examples:

1. Algebraic sets. These are called Affine Algebraic Varieties
2. M is a differentiable manifold, $k = \mathbb{R}$, $k[U] = \{C^\infty$ functions $U \to \mathbb{R}\}$.
3. X is a SWF, $U \subseteq X$ open subset, then U is a SWF. $\mathscr{O}_U(V) = \mathscr{O}_X(V)$.

Definition 1.14 (Morphism of SWFs)**.** *Let X, Y be SWFs, then a morphism $\varphi : X \to Y$ is a continuous map which pulls back regular functions to regular functions. i.e., if $V \subseteq Y$ is open, $f \in \mathscr{O}_Y(V)$, then $\varphi^*(f) \in \mathscr{O}_X(\varphi^{-1}(V))$, $\varphi^*(f) = f \circ \varphi$.*

Definition 1.15 (Isomorphism)**.** *$\varphi : X \to Y$ is an isomorphism if φ is a morphism and $\exists$ a morphism $\psi : Y \to X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$.*

Exercises

1. The id function of a SWF is a morphism.
2. Compositions of morphisms are morphisms.
3. Let X be any SWF and $Y \subseteq \mathbb{A}^n$ closed, that is, an affine variety. Then $f = (f_1, \ldots, f_n) : X \to Y$, is a morphism iff $f_i \in k[X]$ for all i.

Example: $\mathbb{A}^1 \setminus \{0\}$ is (isomorphic to) an affine variety defined by $V(1 - xy)$.
Localization

Let R be a ring and $S \subseteq R$ multiplicatively closed subset. That is, $s, t \in S \Rightarrow st \in S$ and $1 \in S$.

We define $S^{-1}R = \{f/s : f \in R, s \in S\}$. We consider f/s to be the same element as g/t iff there exists $u \in S$ such that $u(ft - sg) = 0$. This is a ring by $\frac{f}{s}\frac{g}{t} = \frac{fg}{st}$ and $\frac{f}{s} + \frac{g}{t} = \frac{ft+gs}{st}$.

Exercise: Check these assertions.

Special Case: If $f \in R$, then $R_f = S^{-1}R$ where $S = \{f^n : n \in \mathbb{N}\}$. In fact, $R_f \cong R[y]/(1 - fy)$.

Definition 1.16 (Reduced Ring)**.** *R is a reduced ring iff $f^n = 0$ implies $f = 0$ for all $f \in R$. Equivalently, $(0) = \sqrt{(0)}$.*

Facts:

1. R reduced implies $S^{-1}R$ is reduced
2. R/I reduced iff $I = \sqrt{I}$

Proposition 1.15. *Let $X \subseteq \mathbb{A}^n$ be a closed affine variety and $f \in A(X)$. Then $D(f)$ if an affine variety, with affine coordinate ring $A(X)_f$.*

Proof. Let $I = I(X) \subseteq k[x_1, \ldots, x_n]$ define $J = (I, yf - 1) \subseteq k[x_1, \ldots, x_n, y]$.

Let $\phi : D(f) \to V(J) \subseteq \mathbb{A}^{n+1}$ by $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, f(a_1, \ldots, a_n)^{-1})$. Note: ϕ is an isomorphism.

What remains is to compute the coordinate ring. $A(X)$ is reduced, and so $A(X)_f$ is reduced. $A(X)_f = k[x_1, \ldots, x_n, y]/J$, so J is a radical ideal, so $J = \sqrt{J} = I(V(J))$. Therefore, $k[D(f)] = k[V(J)] = k[x_1, \ldots, x_n, y]/J = A(X)_f$. □

Definition 1.17 (Prevariety)**.** *A prevariety is a space with functions X such that X has a finite open cover $X = U_1 \cup \ldots \cup U_m$ where U_i is an affine variety.*

Example: Any affine variety is a prevariety.

Exercise: Any prevariety is a Nötherian Topological Space

Example: An open subset of a prevariety is a prevariety. This follows from the previous proposition and the fact that principle open sets are a basis for the topology of any affine variety.

Proposition 1.16. *X is a space with functions, $Y \subseteq \mathbb{A}^n$ an affine variety, we have a 1-1 correspondence:*

$\{$morphisms $X \to Y\}$ $\iff$ $\{k$-algebra homomorphism $A(Y) \to k[X]\}$ by $\varphi \iff \varphi^*$

Proof. Note $\phi \mapsto \phi^*$ is a well defined map. Write $A(\mathbb{A}^n) = k[y_1, \dots, y_n]$, then $I(Y) \subseteq k[y_1, \dots, y_n]$. Then $\bar{y_i}$ is the image of y_i in $A(Y)$. Assume that $\alpha : A(Y) \to k[X]$ is a k-algebra homomorphism. We define $\phi : X \to \mathbb{A}^n$ by $\phi(x) = (\phi_1(x), \dots, \phi_n(x))$.

If $f \in I(Y)$ then $f(\bar{y_1}, \dots, \bar{y_n}) = 0$, so $f(\phi_1, \dots, \phi_n) = \alpha(f(\bar{y_1}, \dots, \bar{y_{n)}}) = 0$, and so $\phi(X) \subseteq Y$. Note, $\phi^*(\bar{y_i}) = y_i \circ \phi = \phi_i = \alpha(\bar{y_i})$. Thus, $\phi^* = \alpha$.

If $\phi : X \to Y$ is a morphism, then $\phi_i = y_i \circ \phi = \phi^*(\bar{y_i})$

Thus, ϕ is the morphism that we construct from ϕ^*. □

Corollary 1.17. *Two affine varieties are isomorphic iff their affine coordinate rings are isomorphic as k-algebras*

Exercise: $\mathbb{A}^n \setminus \{(0, \dots, 0)\}$ is not affine for $n \geq 2$.

Proposition 1.18. *We have a one-to-one correspondence between affine varieties and reduced finitely generated k-algebras, up to isomorphism, by $X \mapsto k[X]$.*

Proof. Last time, we proved that two affine varieties are isomorphic iff their coordinate rings are isomorphic. Thus, $X \mapsto k[X]$ is injective.

Let R be a finitely generated reduced k-algebra generated by $r_1, \dots, r_n \in R$. There is a k-alg homomorphism $\phi k[x_1, \dots, x_n] \to R$ by $x_i \mapsto r_i$ which is surjective. Set $I = \ker \phi$ and let $X = V(I) \subseteq \mathbb{A}^n$.

I is radical, as R is reduced, so $k[X] = k[x_1, \dots, x_n]/I \simeq R$. □

Note: Assume $m \subseteq R$ is a maximal ideal, then $\phi : k[x_1, \dots, x_n] \to R$ as in proof, then $M = \phi^{-1}(m)$ is maximal, and $M = (x_1 - a_1, \dots, x_n - a_n)$. $R/m = k[x_1, \dots, x_n]/M = k$.

Canonical Construction

Let R be a finitely generated reduced k-algebra. Then define $\operatorname{Spec} -m(R) = \{m \subseteq R \text{ max ideals}\}$.

The topology will be that the closed sets $V(I) = \{m \supseteq I | I \subseteq R \text{ and ideal }\}$.

Let $f \in R$. We define $f : \operatorname{Spec} -m(R) \to k$ by $f(m)$ =image of f in $R/m = k$. So f is a function from $\operatorname{Spec} -m$ to k.

I.E., $f(m) \in k \subseteq R$ is the unique element such that $f - f(m) \in m$.

Finally, if $U \subseteq \operatorname{Spec} -m$ is open, $f : U \to k$ is some function, then f is regular if f is locally of the form $f(m) = p(m)/q(m)$ where $p, q \in R$.

Exercise: $\operatorname{Spec} -m(R) \cong X$, where X is the affine variety with coordinate ring R, as spaces with functions.

Subspaces of SWFs

Let X be any space with functions, and $Y \subseteq X$ any subset. Then give Y an "inherited" SWF structure as follows:

We give Y the subspace topology, and if $U \subseteq Y$ is open and $f : U \to k$ is a function, then f is regular iff f can be locally extended to a regular function on

X. That is, for every point $y \in U$, there is an open subset $U' \subseteq X$ containing y and $F \in \mathscr{O}_X(U')$ such that $f(x) = F(x)$ for all $x \in U \cap U'$.

Exercises

1. Y is a SWF
2. $i : Y \to X$ the inclusion map is a morphism.
3. Let Z be a SWF, $\phi : Z \to Y$ function. Then ϕ is a morphism iff $i \circ \phi$ is a morphism.
4. The SWF structure on Y is uniquely determined by (2) and (3) together.
5. Let $Z \subseteq Y \subseteq X$. Then Z inherits the same structure from Y and X.

Example: $X \subset \mathbb{A}^n$ an algebraic set inherits structure from $\mathbb{A}^n$. If $Y \subseteq X$ is closed, then Y inherits structure from X (or $\mathbb{A}^n$).

Proposition 1.19. *A closed subset of a prevariety is a prevariety.*

Proof. Let X be a prevariety, and $Y \subseteq X$ a closed subset. $X = U_1 \cup \ldots \cup U_m$ where U_i are open affine subsets of X.

$U_i \cap Y$ is a closed subset of U_i, which implies that $U_i \cap Y$ is affine, and so Y has the open cover $(U_1 \cap Y) \cup \ldots \cup (U_n \cap Y)$, and so is a prevariety. □

Projective Space

Theorem 1.20. *Two distinct lines in the place intersect in exactly one point. (except when parallel)*

Theorem 1.21. *A line meets a parabola in exactly two points. (except when false)*

We need projective space to remove the bad cases.

Definition 1.18 (Projective Space)**.** *Define an equivalence relation on* $\mathbb{A}^{n+1} \setminus \{0\}$ *by*

$(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ *where* $\lambda \in k^* = k \setminus \{0\}$.

So $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\sim$ *and* $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ *the projection.*

There is a topology on $\mathbb{P}^n$ given by $U \subseteq \mathbb{P}^n$ is open iff $\pi^{-1}(U) \subseteq \mathbb{A}^{n+1}$ is open.

The regular functions on $\mathbb{P}^n$ are $f : U \to k$ such that $\pi^*(f) = \pi \circ f : \pi^{-1}(U) \to k$ is regular.

Thus, $\mathbb{P}^n$ is a SWF called Projective Space. Note: $\mathbb{P}^n = \{$lines through the origin in $\mathbb{A}^{n+1}\}$, and this method of thinking is often very helpful.

We will use the notation $(a_0 : \ldots : a_n)$ for the image $\pi(a_0, \ldots, a_n) \in \mathbb{P}^n$. If $f \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial of total degree d, then $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n)$. Thus it is well-defined to ask if $f(a_0 : \ldots : a_n) = 0$ or not.

Definition 1.19. $D_+(f) = \{(a_0 : \ldots : a_n) \in \mathbb{P}^n : f(a_0 : \ldots : a_n) \neq 0\}$.

Theorem 1.22. *$\mathbb{P}^n$ is a prevariety.*

Proof. Let $U_i = D_+(x_i) \subseteq \mathbb{P}^n$ for $0 \leq i \leq n$.
Claim: $U_i \simeq \mathbb{A}^n$.

$$\phi : \mathbb{A}^n \to U_i : (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \mapsto (a_0 : \ldots : a_{i-1} : 1 : a_{i+1} : \ldots : a_n)$$

$$\psi : U_i \to \mathbb{A}^n : (a_0 : \ldots : a_n) \mapsto \left(\frac{a_0}{a_i}, \ldots, \frac{\hat{a}_i}{a_i}, \ldots, \frac{a_n}{a_i}\right)$$

□

Note:$\mathbb{P}^n = D_+(x_0) \coprod V_+(x_0) = \mathbb{A}^n \coprod \mathbb{P}^{n-1}$, that is, $\mathbb{A}^n = \{(1 : a_1 : \ldots a_n)\}$ the usual n-space and $\mathbb{P}^{n-1} = \{0 : a_1 : \ldots : a_n\}$ points at ∞.

The points at ∞ correspond to lines through the origin in $\mathbb{A}^n$, that is, we can think of them as being directions.

Example: $\mathbb{P}^2$ has "homogeneous coordinate ring" $k[x, y, z]$. We can think of that as $\mathbb{A}^2 = D_+(z) \subset \mathbb{P}^2$ and we know that $k[\mathbb{A}^2] = k[x/z, y/z]$. We want to intersect a parabola with a line.

The vertical line is $\overline{V(x/z - 1)} = V_+(x-z)$ and the parabola is $\overline{V(y/z - (x/z)^2)} = V_+(yz - x^2)$.

And so, $V_+(x - z) \cap V_+(yz - x^2) = \{(1 : 1 : 1), (0 : 1 : 0)\}$, where $(0 : 1 : 0)$ is the point at infinity in the direction "up".

Exercise: $k[\mathbb{P}^n] = k$.

Exercise: X a SWF, $\phi : \mathbb{P}^n \to X$ a function, then ϕ is a morphism iff $\phi \circ \pi : \mathbb{A}^{n+1} \setminus \{0\} \to X$ is a morphism.

Definition 1.20 (Projective Coordinate Ring of $\mathbb{P}^n$)**.** *We define $k[x_0, x_1, \ldots, x_n]$ to be the coordinate ring of $\mathbb{P}^n$. An ideal $I \subseteq k[x_0, \ldots, x_n]$ is homogeneous if it is generated by homogeneous polynomials. Equivalently, $f \in I$ iff each homogeneous component is in I.*

Definition 1.21. *If $W \subseteq \mathbb{P}^n$ is a subset, then $I(W) = I(\pi^{-1}(W)) \subseteq k[x_0, \ldots, x_n]$. Notice that $I(W)$ is homogeneous. Let $f = f_0 + \ldots + f_d \in I(W)$, f_i a form of degree i, then $(a_0 : \ldots : a_n) \in W$, so $0 = f(\lambda a_0, \ldots, \lambda a_n) = f_0(a_0, \ldots, a_n) + \ldots + \lambda^d f_d(a_0, \ldots, a_n)$. As this is true for all λ, $f_i(a_0, \ldots, a_n) = 0$ for all i, and so $f_i \in I(W)$.*

Definition 1.22. *If $I \subseteq k[x_0, \ldots, x_n]$ is a homogeneous ideal, then define $V_+(I) = \{(a_0 : \ldots, a_n) \in \mathbb{P}^n : f(a_0, \ldots, a_n) = 0$ for all $f \in I\}$*

Theorem 1.23 (Projective Nullstellensatz)**.** *If $I \subseteq k[x_0, \ldots, x_n]$ is a homogeneous ideal, then*

1. *$V_+(I) = \emptyset \Rightarrow (x_0, \ldots, x_n)^N \subseteq I$ for some $N > 0$. That is, $\sqrt{I} = (1)$ or $(x_0, \ldots, x_n)$.*

2. $V_+(I) \neq \emptyset$ *then* $I(V_+(I)) = \sqrt{I}$.

Proof. 1. $V_+(I) = \emptyset \iff V(I) = \emptyset$ or $V(I) = \{0\}$. By the regular nullstellensatz, $\sqrt{I} = I(V(I)) = (1)$ or $(x_0, \ldots, x_n)$.

2. $V_+(I) \neq \emptyset$. Then $\overline{\pi^{-1}(V_+(I))} = \pi^{-1}(V_+(I)) \cup \{0\} = V(I) \subseteq \mathbb{A}^{n+1}$. So $I(V_+(I)) = I(V(I)) = \sqrt{I}$.

□

This gives us a 1-1 correspondence between closed subsets of $\mathbb{P}^n$ and radical homogeneous ideals in $k[x_0, \ldots, x_n]$ except for $(x_0, \ldots, x_n)$. This ideal is often called the irrelevant ideal.

Definition 1.23 (Locally Closed subset)**.** *X is a topological space, $W \subseteq X$ a subset is locally closed if it is the intersection of an open set in X and a closed set in X.*

Note: A locally closed subset of a prevariety is a prevariety.

Terminology: a projective variety is any closed subset of $\mathbb{P}^n$ considered as a space with functions. A Quasi-projective variety is a locally closed subset of $\mathbb{P}^n$. An affine variety is a closed subset of $\mathbb{A}^n$. A quasi-affine variety is a locally closed subset of $\mathbb{A}^n$.

We notice that anything affine is also quasi-affine and anything quasi-affine is quasi-projective. Something that is projective will also be quasi-projective.

Exercise: $\mathbb{P}^n$ is not quasi-affine for $n \geq 1$. Later: If X is both projective and quasi-affine, then X is finite.

Definition 1.24 (Projective Coordinate Ring)**.** *$X \subseteq \mathbb{P}^n$ is a closed projective variety, then the projective coordinate ring of $X = k[x_0, \ldots, x_n]/I(X)$.*

Warning: This definition depends on the embedding of X in $\mathbb{P}^n$.

Example: $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ by $\phi(a : b) = (a^2 : ab : b^2)$. This is a morphism. In fact, it is an isomorphism of $\mathbb{P}^1$ and $V_+(xz - y^2)$, but the coordinate ring of $\mathbb{P}^1$ is $k[s, t]$ and the coordinate ring of $V_+(xz - y^2)$ is $k[x, y, z]/(xz - y^2)$. These two rings are NOT isomorphic as k-algebras.

Definition 1.25 (Projective Closure of an affine variety)**.** *$X \subseteq \mathbb{A}^n$ is affine. Then we know that $\mathbb{A}^n = D_+(x_0) \subseteq \mathbb{P}^n$, and $X \subset \mathbb{A}^n \subseteq \mathbb{P}^n$ makes X a quasi-projective variety, so we take $\bar{X}$ = the closure of X in $\mathbb{P}^n$.*

$I = I(X) \subseteq k[x_1, \ldots, x_n]$ if $f = f_0 + \ldots + f_d \in k[x_1, \ldots, x_n]$ where f_i is a form of degree i. Then we define $f^* = x_0^d f_0 + x_0^{d-1} f_1 + \ldots + f_d \in k[x_0, \ldots, x_n]$. And I^* is the ideal generated by $\{f^* : f \in I\}$ in $k[x_0, \ldots, x_n]$.

Exercise: $I(\bar{X}) = I(X)^*$.

Example: $I = (y - x^2, z - x^2) \subseteq k[x, y, z]$. Then $X = V(I) \subseteq \mathbb{A}^3 = D_+(w) \subset \mathbb{P}^n$. $I(\overline{X}) = I^* = (yw - x^2, y - z) \supseteq (wy - x^2, wx - x^2)$

So $V_+(wy - x^2, wz - x^2) = \bar{X} \cup V_+(x, w)$.

We now recall that a graded ring is a ring R with decomposition $R = \oplus_{d \geq 0} R_d$ as an abelian group such that $R_d \cdot R_e \subseteq R_{d+e}$.

e.g., $R = k[x_0, \ldots, x_n]$. $f \in R_d \Rightarrow R_f$ is a $\mathbb{Z}$-graded ring $g \in R_p$ implies that $g/f^m \in R_f$ is homogeneous of degree $p - md$.

Definition 1.26. $R_{(f)} = \{\textit{homogeneous elements of degree zero}\} = (R_f)_0 = \{g/f^m : g \in R_{dm}\}$.

Exercise: $f \in k[x_0, \ldots, x_n]$ homogeneous implies that $k[x_0, \ldots, x_n]_{(f)}$ is a finitely generated reduced k-algebra.

Theorem 1.24. *$f \notin k \Rightarrow D_+(f) \subseteq \mathbb{P}^n$ is affine and in fact $k[D_+(f)] = k[x_0, \ldots, x_n]_{(f)}$.*

Proof. $k[D_+(f)] = \{h \in k[D(f)] : h(\lambda x) = h(x), \forall \lambda \in k^*, x \in D(f)\}$.

If $h \in k[D(f)] = k[x_0, \ldots, x_n]_f$, $h = g/f^m$, $g \in k[x_0, \ldots, x_n]$.

$$\frac{g}{f^m}(\lambda a_0, \ldots, \lambda a_n) = \frac{g}{f^m}(a_0, \ldots, a_n) \iff g \texttt{ homogeneous of degree } md$$

Therefore, $k[D_+(f)] = k[x_0, \ldots, x_n]_{(f)}$.

The identity map $k[D_+(f)] \to k[D_+(f)]$ gives a morphism $\phi : D_+(f) \to \text{Spec}\,-m(k[D_+(f)])$ by $\phi(x) = M_x$ where $M_x = I(\{x\}) \subseteq k[D_+(f)]$

Observe that if $x, y \in D_+(f)$, $x \neq y$ then $\exists$ homogeneous $g \in k[x_0, \ldots, x_n]$ such that $\deg(g) = d$ and $g(x) = 0$ with $g(y) \neq 0$. So $M_x \neq M_y$. $\frac{g}{f} \in M_x, \notin M_y$. Thus, ϕ is injective. Set $h_i = \frac{x_i^d}{f} \in k[D_+(f)]$ for $0 \leq i \leq n$.

$U_i = D(h_i) \subseteq D_+(f)$, $V_i = D(h_i) \subseteq \text{Spec}\,-m(k[D_+(f)]) = \{m \not\ni h_i\}$.

Now we must check that $D_+(f) = \cup_{i=0}^n U_i$ and $\text{Spec}\,-m(k[D_+(f)]) = \cup_{i=0}^n V_i$.

It is enough to prove that $\phi : U_i \to V_i$ is an isomorphism for all i.

$D_+(x_i) \subseteq \mathbb{P}^n$ is affine. $k[D_+(x_i)] = k[x_0/x_i, \ldots, x_n/x_i] = k[x_0, \ldots, x_n]_{(x_i)}$.

Thus, $U_i = D(f/x_i^d) \subseteq D_+(x_i)$ is affine, so $k[U_i] = (k[x_0, \ldots, x_n]_{(x_i)})_{f/x_i^d}$, which is $k[x_0, \ldots, x_n]_{(x,f)} = k[D_+(f)]_{h_i} = k[V_i]$. Thus, $U_i \simeq V_i$. ∎

Example: $f = xz - y^2 \in k[x, y, z]$. $X = D_+(f) \subseteq \mathbb{P}^2$, $R = k[x, y, z]_{(f)}$. R is generated by $A = x^2/f$, $B = y^2/f$, $C = z^2/f$, $D = xy/f$, $E = yz/f$, and $F = xz/f$.

So $X \simeq V(AB - D^2, AC - F^2, BC - E^2, F - B - 1) \subseteq \mathbb{A}^6$ by $(x : y : z) \mapsto (A, B, C, D, E, F)$.

Exercise: $X \subseteq \mathbb{P}^n$ a projective variety $f \in R = k[x_0, \ldots, x_n]/I(X)$ is homogeneous, then $D_+(f) \subseteq X$ is affine with affine coordinate ring $k[D_+(f)] = R_{(f)}$.

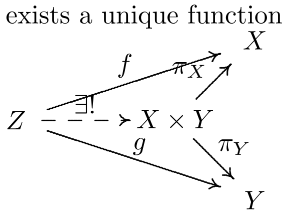
2 Algebraic Varieties

Products

Let X, Y be two sets. Then $X \times Y$, the cartesian product, is the set $\{(x, y) : x \in X, y \in Y\}$.

What is $X \times Y$, really? Well, it is a set with projection $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. This set with the projections satisfies a universal property in the category of sets.

For any set Z with arbitrary functions $f : Z \to X$ and $g : Z \to y$, there exists a unique function $\phi : Z \to X \times Y$ such that $f = \pi_X \circ \phi$ and $g = \pi_Y \circ \phi$.



Definition 2.1 (Product of SWFs)**.** *Let X, Y be spaces with functions. A product of X and Y is a SWF called $X \times Y$ with morphism $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ which satisfies the above universal property except with "morphisms" rather than "functions".*

Exercise: Assume that (P, π_X, π_Y) and (P', π'_X, π'_Y) are two products of X and Y. Then they are isomorphic by unique isomorphism. (See homework problem)

Example: $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$. NOTE: $\mathbb{A}^2$ does not have the product topology!

General Construction: X, Y spaces with functions. Then $X \times Y = \{(x, y) : x \in X, y \in Y\}$ is the point set. If $U \subset X$ and $V \subset Y$ are open, then $U \times V \subset X \times Y$ is open, as it is $\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$.

Let $g_1, \dots, g_u \in \mathscr{O}_X(U)$ and $h_1, \dots, h_n \in \mathscr{O}_Y(V)$ set $f(u, v) = \sum_{i=1}^n g_i(u)h_i(v)$. Then $f : U \times V \to k$ must be regular. Thus, $D_{U\times V}(f) = \{(u, v) \in U \times V : f(u, v) \neq 0\}$ must be open in $X \times Y$. And so, we define our topology by $S \subseteq X \times Y$ is open iff it is a union of sets $D_{U\times V}(f)$. The regular functions $F : S \to k$ are the functions that can locally be written as $f'(u, v)/f(u, v)$ on some $D_{U\times V}(f)$.

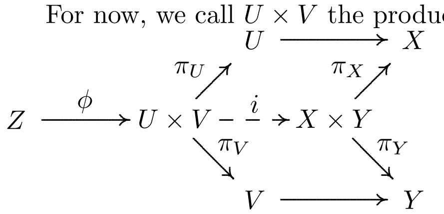
That is, $\forall (x, y) \in S$ $\exists U \subseteq X$ open and $V \subseteq Y$ open and $f(u, v) = \sum_{i=1}^n g_i(u)h_i(v)$ and $f'(u, v) = \sum_{j=1}^m g'_j(u)h'_j(u)$ with $g_i, g'_j \in \mathscr{O}_X(U)$ and $h_i, h'_j \in \mathscr{O}_Y(V)$ such that $(x, y) \in D_{U\times V}(f) \subseteq S$ and $F(u, v) = f'(u, v)/f(u, v)$ for all $(u, v) \in D_{U\times V}(f)$.

Exercises (for $X \times Y$ above)

1. $X \times Y$ is an SWF
2. $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are morphisms.
3. $X \times Y$ is the product of X and Y.

Remark: X, Y SWFs, and $U \subseteq X$, $V \subseteq Y$ are arbitrary subsets, U, V have inherited structure as SWFs. Then $U \times V$ has the product space with functions structure and subspace SWF structure $U \times V \subseteq X \times Y$. These are in fact the same, due to the universal properties.

For now, we call $U \times V$ the product. We obtain the following diagram.



ϕ is a morphism iff $i \circ \phi$ is one, and so we see that the two structures are the same.

Definition 2.2 (Separated SWF)**.** *A SWF X is separated if $\forall$ SWFs Y and morphisms $f, g : Y \to X$ the set $\{y \in Y : f(y) = g(y)\} \subseteq Y$ is closed.*

Example: Let $X = (\mathbb{A}^1 \setminus \{0\}) \cup \{O_1, O_2\}$. We can define $\phi_i : \mathbb{A}^1 \to X$ by taking $a \mapsto \begin{cases} a & a \neq 0 \\ O_i & a = 0 \end{cases}$

We define a topology by $U \subseteq X$ is open iff $\phi_i^{-1}(U) \subseteq \mathbb{A}^1$ is open for all i. A function $f : U \to k$ is regular iff $\phi_i^*(f) = f \circ \phi_i : \phi_i^{-1}(U) \to k$ is regular for all i.

X is a prevariety as $X = \phi_1(\mathbb{A}^1) \cup \phi_2(\mathbb{A}^1)$ and $\phi_i(\mathbb{A}^1) \simeq \mathbb{A}^1$. However, it is not separated, as $\{a \in \mathbb{A}^1 : \phi_1(a) = \phi_2(a)\} = \mathbb{A}^1 \setminus \{0\}$ is not closed in $\mathbb{A}^1$.

Definition 2.3 (Algebraic Variety)**.** *An algebraic variety is a separated prevariety.*

Exercise:

1. Any subspace of a separated SWF is separated.
2. A product of separated SWFs is separated.

Remark: If X is any SWF, then $\Delta : X \to X \times X : x \mapsto (x, x)$ is a morphism. Now we set $\Delta_X = \Delta(X) \subseteq X \times X$. Then $\Delta : X \to \Delta_X$ is an isomorphism.

Lemma 2.1. *X is separated iff $\Delta_X \subseteq X \times X$ is closed.*

Proof. $\Rightarrow$: $\pi_i : X \times X \to X$ be the projections. $\Delta_X = \{z \in X \times X : \pi_1(z) = \pi_2(z)\}$ is closed.

$\Leftarrow$: Let Y be a SWF, $f, g : Y \to X$ maps. Define $\phi : Y \to X \times X$ by $\phi(y) = (f(y), g(y))$ is a morphism. Now $\{y \in Y : f(y) = g(y)\} \Rightarrow \phi^{-1}(\Delta_X)$ is closed. □

Exercise: A topological space X is Hausdorff iff $\Delta_X \subseteq X \times X$ is closed.
NB: Product topology!
Exercise: $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$.

Proposition 2.2. *All affine varieties are varieties.*

Proof. Enough to show that $\mathbb{A}^n$ itself is separated.

$\Delta_{\mathbb{A}^n} \subseteq \mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$ is closed, as $k[\mathbb{A}^{2n}] = [x_1, \ldots, x_n, y_1, \ldots, y_n]$, so $\Delta_{\mathbb{A}^n} = V(\{x_i - y_i\})$. □

Products of Affine Varieties

Lemma 2.3. *Let A, B be finitely generated reduced k-algebras. Then $A \otimes_k B$ is a finitely generated reducted k-algebra and k algebraically closed.*

Recall: ring structure on $A \otimes B$ by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

Proof. If A is generated by $a_1, \dots, a_n$ and B is generated by $b_1, \dots, b_m$ then $A \otimes B$ is generated by $a_1 \otimes 1, \dots, a_n \otimes 1, 1 \otimes b_1, \dots, 1 \otimes b_m$. For example, $k[x_1, \dots, x_n] \otimes k[y_1, \dots, y_m] = k[x_1, \dots, x_n, y_1, \dots, y_m]$.

Let X, Y be affine varieties such that $k[X] = A, k[Y] = B$. We define $\phi : A \otimes_k B \to$the set of all functions $X \times Y \to k$ by $f \otimes g \mapsto \big[(x, y) \mapsto f(x)g(x)\big]$ is a k-algebra homomorphism.

ϕ is injective (which implies that $A \otimes B$ is reduced): Suppose $\phi\left(\sum_{i=1}^n f_i \otimes g_i\right) = 0$. WLOG we can assume $g_1, \dots, g_n$ are linearly independent. Let $x \in X$. Then $\sum_{i=1}^n f_i(x) g_i = 0 \in B$, but $\{g_i\}$ is linearly independent so $f_i(x) = 0$ for all i and x, thus $f_i = 0 \in A$. □

Theorem 2.4. *1. If X, Y are affine then $X \times Y$ is affine and $k[X \times Y] = k[X] \otimes_k k[Y]$.*

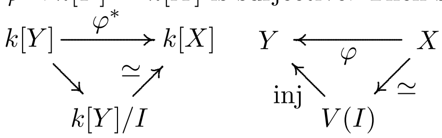
2. A product of prevarieties is a prevariety.

Proof. 1 $\Rightarrow$ 2: X, Y prevarieties, $X = \cup U_i$, $Y = \cup V_j$ with U_i, V_j affine. Then $X \times Y = \cup_{i,j} U_i \times V_j$ affine.

Now, we must only prove 1. Set $P = \text{Spec-}m(k[X] \otimes k[Y])$, We have k-algebra homomorphisms $k[X] \to k[X] \otimes k[Y] : f \mapsto f \otimes 1$ and $k[Y] \to k[X] \otimes k[Y] : g \mapsto 1 \otimes g$. These give morphism $\pi_X : P \to X$ and $\pi_Y : P \to Y$. Claim: $P = X \times Y$. Let Z be a SWF, $p : Z \to X$ and $q : Z \to Y$.

Define k-alg homomorphism $k[X] \mapsto k[Y] \to k[Z]$ by $f \otimes g \mapsto p^*(f) q^*(g)$. This gives us a morphism $\phi : Z \to P$ demonstrating that P is the product. □

Remark: If Y is affine and $X \subseteq Y$ is closed, then $k[X] \simeq k[Y]/I(X)$. On the other hand, assume X, Y are affine, $\varphi : X \to Y$ is a morphism and $\varphi^* : k[Y] \to k[X]$ is surjective. Then set $I = \ker(\varphi^*) \subseteq k[Y]$, we get



Therefore φ is an embedding of X as a closed subset of Y.

Recall: $\Delta : X \to X \times X : x \mapsto (x, x)$ gives $X \simeq \Delta_X = \Delta(X) = \{(x, x) : x \in X\}$.

Proposition 2.5. *A prevariety X is separated iff $\forall$ open affine $U, V \subseteq X$, $U \cap V$ is affine and $k[U \times V] = k[U] \otimes k[V] \to k[U \cap V] = k[\Delta_{U \cap V}]$ is surjective.*

Proof. $\Rightarrow$: $U \cap V \simeq \Delta_{U \cap V} = \Delta_X \cap (U \times V) \subseteq U \times V$ is closed, thus $U \cap V$ is affine and $k[U \times V] \to k[U \cap V]$ is surjective.

$\Leftarrow$: If $U, V, U \cap V$ are affine and $k[U \times V] \to k[U \cap V]$ is surjective, then $\Delta : U \cap V \to U \times V$ is an inclusion of closed subsets. Thus, $\Delta_X \cap (U \times V) \subseteq U \times V$ closed. So if $X \times X = \cup U \times V$ is an open cover then $\Delta_X \subseteq X \times X$ is closed. □

Exercises

1. X is a prevariety such that $\forall x, y \in X$ there is an open affine $U \subseteq X$ such that $x, y \in U$. Then X is separated.

2. $\mathbb{P}^n$ has this property.

Corollary 2.6. *Quasi-projective varieties are separated.*

We want to show that the products of projective varieties are again projective.

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are closed. Then $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed. It is enough to show that $\mathbb{P}^n \times \mathbb{P}^m$ is projective, that is, $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^N$ is closed.

Segre Map: If $N = (n+1)(m+1) - 1 = nm + n + m$, then we can define $s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N : (x_0 : \ldots : x_n) \times (y_0 : \ldots : y_m) \mapsto (x_0y_0 : x_0y_1 : \ldots : x_0y_m : x_1y_0 : \ldots : x_ny_m)$.

We call the projective coordinates on $\mathbb{P}^N$ as z_{ij} for $0 \leq i \leq n, 0 \leq j \leq m$.

Exercise: $s : \mathbb{P}^n \times \mathbb{P}^m \to V_+(\{z_{ij}z_{pq} - z_{iq}z_{pj}\}) \subseteq \mathbb{P}^N$

Note: $\mathbb{P}^n \times \mathbb{P}^n \subseteq \mathbb{P}^{n^2+2n}$ is closed.

Exercise: $\Delta_{\mathbb{P}^n} = V_+(\{z_{ij} - z_{ji}\}) \subseteq \mathbb{P}^N$.

Complete Varieties: analogues of compact manifolds

Definition 2.4 (Complete)**.** *A variety X is complete if for any variety Y, the projection $\pi_Y : X \times Y \to Y$ is closed. (i.e.: $Z \subseteq X \times Y$ is closed implies that $\pi_Y(Z) \subseteq Y$ is closed)*

Note: 1) closed subsets of complete varieties are complete.

2) Products of complete varieties are complete.

Examples:Points are complete.

$\mathbb{A}^1$ is not complete, as $Z = V(xy-1) \subseteq \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2 = X \times Y$ is such that $\pi_Y(Z)$ is not closed, as it is $\mathbb{A}^1 \setminus \{0\}$

Proposition 2.7. *Let $\varphi : X \to Y$ be a morphism of varieties. If X is complete, then $\varphi(X)$ is closed in Y and is complete.*

Proof. $\Gamma(\varphi) = \{(x, \varphi(x)) \in X \times Y : x \in X\} \subseteq X \times Y = (\varphi \times 1)^{-1}(\Delta Y)$

As Y is separated, $\Gamma(\varphi) \subseteq X \times Y$ is closed. X is complete implies that $\varphi(X) = \pi_Y(\Gamma(\varphi)) \subseteq Y$ is closed.

Now, let $Z \subseteq \varphi(X) \times Y'$ be closed. Then

$$\begin{array}{ccc} X \times Y' & \xrightarrow{\varphi \times 1} & \varphi(X) \times Y' \\ & \searrow^{\tilde{\pi}_{Y'}} & \downarrow \pi_{Y'} \\ & & Y' \end{array}$$

$W = (\varphi \times 1)^{-1}(Z) \subseteq X \times Y'$ is closed, $\pi_{Y'}(Z) = \pi_{Y'}((\varphi \times 1)(W)) = \tilde{\pi}_{Y'}(W)$ is closed. □

Exercise: $\varphi : X \to Y$ is a continuous map of topological spaces then X is irreducible implies that $\varphi(X)$ is irreducible.

Proposition 2.8. *If X is an irreducible complete variety then $k[X] = k$.*

Proof. Let $f \in k[X]$. $f : X \to \mathbb{A}^1$ a morphism. As X is complete, $f(X)$ must be is irreducible, closed and complete. Thus, it must be a point. Thus, f is constant. □

Proposition 2.9. *A complete quasi-affine variety is finite.*

Proof. X is such a variety, without loss of generality X is irreducible. $X \subseteq \mathbb{A}^n$ is locally closed, then $x_i : X \subseteq \mathbb{A}^n \to k$ must be constant, and so x is a point. □

Theorem 2.10. $\mathbb{P}^n$ *is complete.*

Note: $I \subseteq S = k[x_0, \ldots, x_n]$ a homogeneous ideal, then $V_+(I)$ is not empty iff $I_d \subsetneq S_d$ for all $d \in \mathbb{N}$.

Proof. Let Y be a variety and $Z \subseteq \mathbb{P}^n \times Y$ be closed. Show that $\pi_Y(Z) \subseteq Y$ is closed.

$Y = \cup Y_i$ and open affine cover. It is enough to show that $\pi_Y(Z) \cap Y_i \subseteq Y_i$ is closed, that is, $\pi_{Y_i}(Z \cap (\mathbb{P}^n \times Y_i))$ is closed. So we take $\pi_{Y_i} : \mathbb{P}^n \times Y_i \to Y_i$.

Thus, WLOG, we assume Y is affine. Then let $C(Z) = (\pi \times \mathrm{id})^{-1}(Z) \subseteq \mathbb{A}^{n+1} \times Y$. $k[\mathbb{A}^{n+1} \times Y] = S \otimes k[Y] = k[Y][x_0, \ldots, x_n] = \oplus_{d \geq 0} S_d \otimes_k k[Y]$, so it is a graded ring.

Note that $(y, (a_0, \ldots, a_n)) \in C(Z)$ implies that $(y, (\lambda a_0, \ldots, \lambda a_n)) \in C(Z)$ for $\lambda \in k$.

Thus, $I(C(Z)) \subseteq k[\mathbb{A}^{n+1} \times Y]$ is a homogeneous ideal. We write $I(C(Z)) = (f_1, \ldots, f_m)$ with $f_i \in S_{d_i} \otimes k[Y]$.

For $y \in Y$, $f_i(y) = f_i(-, y) \in S_{d_i}$. We observe that $y \in \pi_Y(Z)$ iff $\exists x \in \mathbb{P}^n$ such that $(x, y) \in Z$. This happens iff $V_+(f_1(y), \ldots, f_m(y)) \neq \emptyset \subseteq \mathbb{P}^n$. This is true iff $(f_1(y), \ldots, f_m(y))_d \neq S_d$ for all $d \geq 0$.

Fix $d \geq 0$, then $y \in Y$ defines a linear map $\Phi_Y : \oplus_{i=1}^m S_{d-d_i} \to S_d : (g_1, \ldots, g_m) \mapsto \sum_{i=1}^m f_i(y) g_i$.

Note: Every entry of the matrix Φ_Y is a regular function of Y.

Now, $(f_1(y), \ldots, f_m(y))_d \neq S_d$ iff $\mathrm{rank}(\Phi_Y) < \dim(S_d) = \binom{n+d}{n}$ which holds iff all minors in Φ_Y of size $\binom{n+d}{n}$ vanish. Therefore, $W_d = \{y \in Y : (d_1(y), \ldots, f_m(y))_d \neq S_d\} \subseteq Y$ is closed. Finally, $\pi_Y(Z) = \cap_{d \geq 0} W_d$, and so is closed. □

Challenge: Find a complete variety that is not projective.
Exercise:

1. Let X be a topological space and $W \subseteq X$ is a subset. $\overline{W} = X$ iff $W \cap U \neq \emptyset$ for all nonempty open sets $U \subseteq X$.
2. $f : X \to Y$ continuous and $\overline{W} = X$ and $\overline{f(X)} = Y$ then $\overline{f(W)} = Y$
3. X is irreducible and $\emptyset \neq U \subseteq X$ is open. Then $\overline{U} = X$ and U is irreducible.

Rational Map: X and Y are irreducible varieties. The ideal is that a morphism $f : X \to y$ is uniquely determined by restriction to any non-empty open subset of X.

Consider pairs (U, f) where $U \subseteq X$ nonempty and open and $f : U \to Y$ is a morphism. Relation: $(U, f) \sim (V, g)$ iff $f = g$ on $U \cap V$ because Y is separated and X is irreducible, this is an equivalence relation. Checking this is an exercise.

Definition 2.5 (Rational Map)**.** *A rational map $f : X \dashrightarrow Y$ is an equivalence class for $\sim$.*

X irreducible implies that $U \cap W \supseteq U \cap V \cap W$ dense, and Y separated implies that $f = h$ on a closed subset of $U \cap W$.

Remark: If $f : X \dashrightarrow Y$ then there is a maximal open $U \subseteq X$ where f is defined as a morphism. $U = \cup_{(V,g)\sim f} V \subset X$.

Example: $f : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2 : (x, y) \mapsto (x/y, y/x^2)$ defined as a morphism of $D(xy)$.

Exercise: $f : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2 \subseteq \mathbb{P}^2$. Find the max open where $f : \mathbb{A}^2 \dashrightarrow \mathbb{P}^2$ is defined.

Definition 2.6 (Rational Function)**.** *A rational function on X is a rational map $f : X \dashrightarrow \mathbb{A}^1 = k$. f is given by a regular function $f : U \to k$ where $\emptyset \neq U \subseteq X$ open.*

$k(X) = \{f : X \dashrightarrow k\}$ is the field of rational functions on X.

Note: If $(U, f), (V, g) \in k(X)$ then $f + g, f - g, fg : U \cap V \to k$ define rational functions on X. If $f \neq 0$ in $k[U]$ then $\emptyset \neq D(f) \subseteq U$ is open, and $1/f : D(f) \to k$ is regular. Thus, $1/f = (D(f), 1/f) \in k(X)$.

Examples: $k(\mathbb{A}^n) = k(x_1, \ldots, x_n)$. $k(\mathbb{P}^n) = k(x_1/x_0, \ldots, x_n/x_0)$.

Proposition 2.11. *Let X be an irreducible variety*

1. *If $\emptyset \neq U \subseteq X$ open, then $k(X) = k(U)$.*
2. *If X is affine, then $k(X) = k[X]_0$ =field of fractions of $k[X]$.*

Proof.

1. $k(X) \to k(U) : (V, g) \to (V \cap U, g|_{V \cap U})$ is isomorphism.
2. Define $k[X]_0 \to k(X) : f/g \mapsto (D(g), f/g)$.

   Injective: As this is a homomorphism of fields, it is enough to say that it is not identically zero, and it maps 1 to $(X, 1)$, which is not the zero function.

   Surjective: If $f : U \to k$ is regular, $\emptyset \neq U \subseteq X$ open. Find $0 \neq g \in k[X]$ such that $\emptyset \neq D(g) \subseteq U$. Then $f \in k[D(g)] = k[X]_g \subseteq k[X]_{x_0}$.

□

Definition 2.7 (Dominant)**.** $(U, f) : X \dashrightarrow Y$ *is dominant if* $\overline{f(U)} = Y$.

Indep. of rep.: If $\emptyset \neq V \subseteq U$ open, then $\overline{f(V)} = Y$ by the homework.

Suppose $(U, f) : X \dashrightarrow Y$ is dominant and $(V, g) : Y \dashrightarrow Z$ is any rational map, then we can compose $g \circ f : X \dashrightarrow Z$, as $\overline{f(U)} = Y$ so $f(U) \cap V \neq \emptyset$, so $f^{-1}(V) \neq \emptyset \subseteq U$.

$g \circ f = (f^{-1}(V), g \circ f)$.

<u>Exercise</u>: If f, g both dominant, then $g \circ f$ dominant.

Proposition 2.12. *X, Y irreducible varieties, then there is a one to one correspondence between $\{\phi : X \to Y$ with ϕ dominant$\}$ and field extensions $k(Y) \subseteq k(X)$ by $\phi \mapsto \phi^* = [h \mapsto h \circ \phi]$*

Proof. WLOG X, Y are affine. The map $\phi \mapsto \phi^*$ is:

Injective: So we let $\psi : X \dashrightarrow Y$ dominant and $\psi^* = \phi^*$. Take $D(h) \subseteq X$ such that ϕ and ψ are both defined on $D(h)$.

$$\begin{array}{ccc} k(Y) & \xrightarrow{\phi^* = \psi^*} & k(X) \\ \subseteq \uparrow & & \subseteq \uparrow \\ k[Y] & \longrightarrow & k[X]_h \end{array}$$

This diagram commutes, and so $\phi = \psi$ on $D(h)$.

Surjective: Let $\alpha : k(Y) \to k(X)$ be a k-algebra homomorphism. $k[Y]$ is generated by $f_1, \ldots, f_n$. $\alpha(f_i) = g_i/h_i$, $g_i, h_i \in k[X]$. Let $h = h_1 \ldots h_n \in k[X]$. So $\alpha : k[Y] \to k[X]_h$ is a k-alg hom. This gives us a morphism $\phi : D(h) \to Y$ and $\phi^* = \alpha$. □

Definition 2.8 (Birational)**.** *Let $f : X \dashrightarrow Y$ be a rational map. It is birational if f is dominant and there is a dominant $g : Y \dashrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ as rational maps.*

Definition 2.9 (Birationally Equivalent)**.** *X and Y are birationally equivalent (often X and Y are birational) written $X \approx Y$ if there exists $f : X \dashrightarrow Y$ a birational map.*

<u>Examples:</u> $\mathbb{A}^2 \approx \mathbb{P}^2 \approx \mathbb{P}^1 \times \mathbb{P}^1$.

If $U, V \subseteq X$ open and X irred., then $U \approx Y$.

Theorem 2.13. *The following are equivalent*

1. $X \approx Y$
2. $k(X) \simeq k(Y)$ *as k-algebras*
3. $\exists \emptyset \neq U \subset X, V \subset Y$ *open such that $U \simeq V$ as varieties.*

Proof. $3 \Rightarrow 2$ is clear from the first prop on this topic.

$2 \Rightarrow 1$ is clear from the second prop on this topic.

$1 \Rightarrow 3$: Let $(U, f) : X \dashrightarrow Y$ and $(V, g) : Y \dashrightarrow X$ be inverses. Set $U_0 = f^{-1}(V) \subseteq U$. Then $g \circ f : U_0 \to V \to X$ must be the inclusion of U_0 into

X. Thus, $g(f(U_0)) \subseteq U_0$, so $f(U_0) \subseteq g^{-1}(U_0)$. Set $V_0 = g^{-1}(U_0) \subseteq V$. Then $U_0 \simeq V_0$ as varieties by f, g. □

Definition 2.10 (Rational Variety)**.** *An irreducible variety is rational if it is birational to $\mathbb{A}^n$ for some n.*

Examples: Any curve $C \subseteq \mathbb{P}^2$ of degree 2 is rational.

Let $C = V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$ is rational. Let $\phi : C \dashrightarrow \mathbb{A}^1$ by $\phi(x, y) = y/x$. The inverse should be $\psi : \mathbb{A}^1 \to C$ by $\psi(t) = (1 - t^2, t - t^3)$. So $\phi \circ \psi(t) = (t - t^3)/(1 - t^2) = t = \mathrm{id}_{\mathbb{A}^1}$, and the opposite is also an identity (Exercise, show this).

Challenge: Show that $E = V(y^2 - x^3 + x) \subseteq \mathbb{A}^2$ is not rational.

Big Challenge: If C is any irreducible variety and there exists a dominant rational map $\mathbb{A}^1 \dashrightarrow C$ then C is rational.

Transcendence Degree

Let $k \subseteq L$ a field extension. Then L is algebraic over k if for all $f \in L$, there is a polynomial equation $f^n + a_1 f^{n-1} + \ldots + a_n = 0$ for $a_i \in k$.

$S \subseteq L$ subseteq, then S is algebraically independent over k if for all $s_1, \ldots, s_n \in S$ with $s_i \neq s_j$ for $i \neq j$, then $k[x_1, \ldots, x_n] \to L$ by $x_i \mapsto s_i$ is injective.

Definition 2.11 (Transcendence Basis)**.** *A transcendence basis for L over k is a set $B \subseteq L$ such that B is algebraically independent over k and L is algebraic over $k(B)$.*

Theorem 2.14.

1. *All transcendence bases have the same cardinality.*
2. *If $S \subseteq \Gamma \subseteq L$ subsets such that S is alg indep over k and L is alg over $k(\Gamma)$, then there exists a transcendence basis B for L over k such that $S \subseteq B \subseteq \Gamma$.*

Proof. The idea is as "any vector space has a basis." □

Exercise: Prove where L is a finitely generated extension of k.

Lang's Algebra contains the proof.

Definition 2.12 (Transcendence Degree)**.** *The transcendence degree* $\operatorname{tr}\deg_k(L) = \operatorname{tr}\deg(L) =$ *the number of elements in any transcendence basis for L over k.*

Definition 2.13 (Dimension)**.** *Let X be an irreducible variety. Then define* $\dim(X) = \operatorname{tr}\deg(k(X))$.

Examples:

1. $\dim(\mathbb{A}^n) = \operatorname{tr}\deg_k k(x_1, \ldots, x_n) = n$
2. If X is irreducible and $\dim(X) = 0$, then $k \subseteq k(X)$ is algebraic extension, then $k(X) = k$. Thus, X is a point.

Some terminology: a curve is a variety of dimension 1, a surface is a variety of dimension 2, and an n-fold is a variety of dimension n.

Notation: If R is a finitely generated domain over k, then we write $\operatorname{tr}\deg(R) = \operatorname{tr}\deg(R_0)$.

We will state the following without proof.

Theorem 2.15 (Principle Ideal Theorem)**.** *If R is a finitely generated domain over k and $0 \neq f \in R$ and $P \subseteq R$ is a minimal prime, then P is a minimal prime containing f. Then* $\operatorname{tr}\deg(R/P) = \operatorname{tr}\deg(R) - 1$.

Geometric Statement: If X is any irreducible variety and $0 \neq f \in k[X]$ and if $Z \subseteq V(f)$ is an irreducible component then $\dim Z = \dim X - 1$.

Proof. Take $U \subseteq X$ open affine such that $U \cap Z \neq \emptyset$. Then $Z \cap U = V(P) \subseteq U$, $P \subseteq k[U]$ prime ideal. Z is a component of $V(f) \Rightarrow P$ is minimum over $(f) \subseteq k[U]$.

Thus, $\dim(Z) = \operatorname{tr}\deg(k[U]/P) = \operatorname{tr}\deg k[U] - 1 = \dim X - 1$. □

Theorem 2.16. *Let X be an irreducible variety, and let $\emptyset \neq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X$ be a maximal chain of irreducible closed subsets. Then* $\dim(X) = n$.

Proof. WLOG, X is affine. Take $0 \neq f \in I(X_{n-1})$. Then $X_{n-1} \subseteq V(f)$ is a component. PIT says that $\dim X_{n-1} = \dim X - 1$.

Induction implies that $\dim X_{n-1} = n - 1$, so $\dim X = n$. □

Definition 2.14. *If X is any variety, set* $\dim(X) =$*the supremum of all n such that $\exists$ a chain $\emptyset \neq X_1 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n \subseteq X$ where $X_i \subseteq$ irreducible and closed for all i.*

Exercises

1. $X = X_1 \cup \ldots \cup X_m$ and $X_i \subseteq X$ closed, then $\dim X = \max \dim(X_i)$
2. $\dim(X \times Y) = \dim X + \dim Y$

Recall: If R is a ring, then $\dim(R)$ is the supremum of all n such that $\exists$ $P_n \subsetneq P_{n-1} \subsetneq \ldots \subsetneq P_0 \subseteq R$ where P_i is a prime ideal.

Note: If X is affine then $\dim X = \dim k[X]$.

Theorem 2.17 (PIT For Several Equations)**.** *If X is an irreducible variety and $f_1, \ldots, f_r \in k[X]$ and $Z \subseteq V(f_1, \ldots, f_r)$ are components, then* $\dim Z \geq \dim X - r$.

Proof. Enough to show that if $W \subseteq X$ is a closed subset and each component of W has $\dim \geq d$, then each component of $W \cap V(f)$ has $\dim \geq d - 1$ for all $f \in k[X]$.

Let $Z \subseteq W$ be a component. If $f|_Z = 0$ then $V(f) \cap Z = Z$. If $f|_Z \neq 0$ then every component of $Z \cap V(f)$ has dim= $\dim(Z) - 1 \geq d - 1$.

Therefore $W \cap V(f) =$ union of finitely many irreducible closed subsets of $\dim \geq d - 1$. □

Lemma 2.18 (Prime Avoidance)**.** *X is an affine variety, $Z \subseteq X$ an irreducible closed subset and $X_1, \ldots, X_m \subseteq X$ are also irreducible closed subsets, then if $X_i \nsubseteq Z$ then $\exists f \in I(Z)$ such that $f \notin I(X_i)$.*

Proof. Induction on m.

If $m = 1$, then $X_1 \nsubseteq Z \Rightarrow I(Z) \nsubseteq I(X_1) \Rightarrow \exists f \in I(Z) \setminus I(X_1)$.

For $m \geq 2$, take $f_i \in I(Z)$ such that $f_i \notin I(X_j)$ for $j \neq i$. If any $f_i \notin I(X_j)$, then done. Take $f = f_i$.

If $f_i \in I(X_i)$ for all i, then $f = f_1 + f_2 f_3 \ldots f_m \in I(Z)$. □

Definition 2.15 (Codimension)**.** *If X is any variety, $Z \subseteq X$ closed and irreducible, let $X_1, \ldots, X_m$ be the components of X containing Z. Set $\operatorname{codim}(Z; X) = \dim(X_1 \cup \ldots \cup X_m) - \dim Z$.*

E.g. X is the union of a line and a plane, Z is a single point of X. Then $\operatorname{codim}(Z; X)$ is 2 if it is a point in the plane, 1 otherwise.

Theorem 2.19 (Reverse PIT)**.** *X affine, $Z \subseteq X$ irreducible closed and $c = \operatorname{codim}(Z; X)$. Then $\exists f_1, \ldots, f_c \in k[X]$ such that $Z \subseteq V(f_1, \ldots, f_c)$ irreducible component.*

Proof. If Z is a component of Z, then $c = 0$.

Otherwise, no components of X are contained in Z, so the lemma implies that there exists $f_1 \in k[X]$ such that $f_1 \in I(Z)$ and f_1 does not vanish on any component of X. PIT implies that $\operatorname{codim}(Z; V(f)) < c$.

Induction on c gives us that there are $f_2, \ldots, f_c \in I(Z)$ such that Z is a component of $V(f_1, \ldots, f_c)$. □

<u>Resultants</u>

Let K be an arbitrary field, and $f(T) = a_n T^n + \ldots + a_1 T + a_0$ and $g(T) = b_m T^m + \ldots + b_1 T + b_0 \in K[T]$.

Q: Do $f(T)$ and $g(T)$ have a common factor?

$$\text{Set } A = \begin{bmatrix} a_n & a_2 & a_1 & a_0 & 0 \\ & a_n & a_2 & a_1 & a_0 \\ b_m & b_1 & b_0 & & \\ & b_m & b_1 & b_0 & \\ & & b_m & b_1 & b_0 \end{bmatrix}.$$

Definition 2.16 (Resultant)**.** *We define $\operatorname{Res}(f, g) = \det A \in K$.*

Let $\vec{v} = (c_{m-1}, c_0, d_{n-1}, d_1, d_0) \in K^{n+m}$, then $\vec{v} \cdot A = (r_{n+m-1}, \ldots, r_1, r_0) \in K^{n+m}$. Then $(c_{m-1}^{m-1} + \ldots + c_1 T + c_0) f(T) + (d_{n-1} T^{n-1} + \ldots + d_1 T + d_0) g(T) = r_{m+n-1} T^{m+n-1} + \ldots + r_1 T + r_0$.

Proposition 2.20. *Suppose $a_n \neq 0$, then $\operatorname{Res}(f, g) \neq 0 \iff (f, g) = 1 \in K[T]$.*

Proof. $\operatorname{Res}(f, g) = 0$ iff $\exists \vec{v} \in K^{m+n}$ such that $\vec{v} \cdot A = 0$ iff $\exists p(T), q(T)$ of deg $\leq m - 1, n - 1$ such that $p(T) f(T) = q(T) g(T)$, iff $(f, g) \neq 1$. □

If $f(T) = \sum_{i=0}^n a_i T^i$ then we allow formal differentiation, that is, $f'(T) = \sum_{i=1}^n i a_i T^{i-1} \in K[T]$.

Note that $(fg)' = f'g + fg'$, and similar rules still hold.

Corollary 2.21. *If $f(T) = a_n T^n + \ldots + a_1 T + a_0$ then $f(T)$ has n different roots in $\overline{K}$ iff $\mathrm{Res}(f, f') \neq 0$.*

Proof. $f(T) = a_n \prod_{i=1}^r (T - \alpha_i)^{d_i}$, where $\alpha_i \neq \alpha_j$ for $i \neq j$. Then $f'(T) = a_n \sum_{i=1}^r d_i (T - \alpha_i)^{d_i - 1} \prod_{j \neq i} (T - \alpha_j)^{d_j}$.

$\mathrm{Res}(f, f') = 0$ iff $(f, f') \neq 1$ iff $f'(\alpha_\ell) = 0$ for some ℓ iff $d_\ell \geq 2$ for some ℓ. □

Definition 2.17 (Discriminant)**.** *The discriminant of $f(T)$ is* $\mathrm{Res}(f, f')$.

Exercise: $a \neq 0$ and $f(T) = aT^2 + bT + c$ then discriminant$= -a(b^2 - 4ac)$.

Remark: If $\mathrm{char}(K) = 0$ and if $f(T) \in k[T]$ is an irreducible polynomial, then $(f(T), f'(T)) = 1$ so $\mathrm{Res}(f, f') \neq 0$. In $\mathrm{char}(K) = p$, then $f'(T)$ may be zero, for example $(T^p + 1)' = 0$.

Remark: $\varphi : X \to Y$ is a morphism, then $\dim(\overline{\varphi(X)}) \leq \dim(X)$. This is as $k(\overline{\varphi(X)}) \subseteq k(X)$.

Theorem 2.22. *$\phi : X \to Y$ is a dominant morphism of irreducible varieties, such that $k(Y) \subseteq k(X)$ is a finite extension of degree d. Suppose that* $\mathrm{char}(k) = 0$ *or $k(X)/k(Y)$ is separable. Then $\exists$ dense open $V \subseteq Y$ such that $|\phi^{-1}(y)| = d$ for all $y \in V$.*

Proof. Assume X, Y affine and $k[X] = k[Y][f]$. Let $P(T) = a_d T^d + \ldots + a_1 T + a_0 \in k(Y)[T]$ be the minimum polynomial for $f \in k(X)$ over $k(Y)$. ie $P(f) = 0 \in k(X)$.

WLOG, $a_i \in k[Y]$ for all i and we can replace Y with $D(a_d)$ and X with $\phi^{-1}(D(a_d)) \subseteq X$. We may assume that $a_d = 1$. Now $k[X] = k[Y][T]/(P(T))$. This implies that $X \simeq V(P) \subseteq Y \times \mathbb{A}^1 \overset{\pi_Y}{\to} Y$, and $\phi : X \to Y$ goes through this path.

If $(y, t) \in Y \times \mathbb{A}^1$ then we set $P_y(T) = \sum_{i=0}^d a_i(y) T^i \in k[T]$. $(y, t) \in X$ iff $P_y(t) = 0$. Let $\Delta = \mathrm{Res}(P, P') \in k[Y]$. $P(T)$ irreducible and $\mathrm{char}(k) = 0$ imply that $\Delta \neq 0$. Note that $\mathrm{Res}(P_y, P_y') = \Delta(y)$.

Thus, if $y \in D(\Delta)$, then $P_y(t) = 0$ has exactly d solutions.

Now the general case: X and Y are irreducible varieties and $\phi : X \to Y$ dominant. Let $V \subseteq Y$ and $U \subseteq \phi^{-1}(V) \subseteq X$ be open affines. Then $\dim(\overline{\phi(X \setminus U)}) \leq \dim(X \setminus U) < \dim(X) = \dim(Y)$. Thus $\overline{\phi(X \setminus U)} \subsetneq Y$, and so $\exists h \in k[V]$ such that $D(h) \cap \phi(X \setminus U) = \emptyset$.

We can replace X with $D(\phi^* h)$ and Y with $D(h)$. And so, WLOG, X, Y affine.

ϕ dominant implies that $k[Y] \subseteq k[X]$ and $k[X]$ generated by $f_1, \ldots, f_n$. Then $k[Y] \subseteq k[Y][f_1] \subseteq \ldots \subseteq k[Y][f_1, \ldots, f_n] = k[X]$ gives $X = X_n \to X_{n-1} \to \ldots \to X_1 \overset{\psi}{\to} Y$ a sequence of dominant maps.

Induction on n: $\exists$ a dense open $U \subseteq X_1$ such that all points of U are hit by $d_1 = [k(X) : k(X_1)]$ pts of X.

As above: $\overline{\psi(X_1 \setminus U)} \subsetneq Y$ implies $\exists h \in k[Y]$ such that $D(h) \cap \psi(X_1 \setminus U) = \emptyset$, and so $\psi^{-1}(D(h)) = D(\phi^* h) \subseteq U$. $\psi : D(\phi^* h) \to D(h)$ gives $k[D(\phi^* h)] = k[X]_{\phi^* h} = k[X]_h$. Thus, $k[Y][f_1]_h = k[Y]_h[f_1] = k[D(h)][f_1]$.

Thus, the first case implies that $\exists \emptyset \neq V \subseteq D(h) \subseteq Y$ open such that $|\psi^{-1}(y)| = [k(X_1) : k(Y)]$. Since $\psi^{-1}(D(h)) \subseteq U$ we have $|\phi^{-1}(y)| = [k(X_1) : k(Y)] \cdot d_1 = d$. ☐

Exercise: $\pi_Y : X \times Y \to Y$ is an open map. That is, if $U \subseteq X \times Y$ is open then $\pi_Y(U)$ is open in Y.

Corollary 2.23. *$\phi : X \to Y$ is a dominant morphism of irreducible varieties. Then $\phi(X)$ contains a dense open subset of Y.*

Proof. We can assume that X, Y are affine. Choose $B = \{f_1, \dots, f_n\} \subseteq k[X]$ such that B is a transcendence basis for $k(X)/k(Y)$.

Then $k[Y] \subseteq k[Y][f_1, \dots, f_n] \subseteq k[X]$ gives $X \xrightarrow{\psi} Y \times \mathbb{A}^n \xrightarrow{\pi_Y} Y$ and ϕ is the composition.

The theorem says that there is a open subset $U \subseteq Y \times \mathbb{A}^n$ such that $U \subseteq \psi(X)$. As π_Y is an open mapping, $\pi_Y(U)$ is open. ☐

Definition 2.18. *Let X be a variety.*

1. *$W \subseteq X$ is locally closed if $W =$ open $\cap$ closed.*
2. *$W \subseteq X$ is constructible if $W =$the union of finitely many locally closed subsets.*

Example: $W = D(xy) \cup \{0\} \subseteq \mathbb{A}^2$. Notice: $\phi : \mathbb{A}^2 \to \mathbb{A}^2 : (x, y) \mapsto (x^2 y, xy)$, then $\phi(\mathbb{A}^2) = W$.

Exercise: $\phi : X \to Y$ is an arbitrary morphism of varieties, then $\phi(X)$ is constructible.

3 Nonsingular Varieties

Local Rings

Definition 3.1 (Local Ring at a point)**.** *If X is an irreducible variety and $x \in X$ then $\mathcal{O}_{X,x} = \{f \in k(X) : f(x) \text{ defined}\}$. This is a local ring with maximal ideal $\mathfrak{m}_x = \{f \in \mathcal{O}_{X,x} : f(x) = 0\}$.*

Note: If $U \subseteq X$ is any open subset, then $k[U] = \bigcap_{x \in U} \mathcal{O}_{X,x} \subseteq k(X)$.

Let $U \subseteq X$ open affine, $x \in U$ then $M = I(\{x\}) \subseteq k[U]$. $f \in \mathcal{O}_{X,x}$ then f is defined on $D(h) \subseteq U$ for some $h \in k[U] \setminus M$, so $f = g/h^n$ where $g \in k[U]$. And so, $\mathcal{O}_{X,x} = \{g/h : g, h \in k[U], h(x) \neq 0\} = k[U]_M$.

Remark: X not irreducible implies that $\mathcal{O}_{X,x} = \varinjlim_{U \ni x} \mathcal{O}_X(U)$. If $U \subseteq X$ open affine, $x \in U$ we still have that $\mathcal{O}_{X,x} = k[U]_{I(\{x\})}$.

Note: If X is irreducible then $\dim(X) = \dim \mathcal{O}_{X,x}$ for any $x \in X$. If we let X be any variety, then $\dim(X) = \max_{x \in X} \dim \mathcal{O}_{X,x}$.

Definition 3.2 (Regular Local Ring)**.** *If $(R, \mathfrak{m})$ is a local ring, then $F = R/\mathfrak{m}$ is a field called the residue field of R and $\mathfrak{m}/\mathfrak{m}^2$ is an F-vector space.*

R is a regular local ring if $\dim_F(\mathfrak{m}/\mathfrak{m}^2) = \dim R$.

Exercise: If R is a Nötherian Local Ring then $\dim_F(\mathfrak{m}/\mathfrak{m}^2)$ =min number of generators for $\mathfrak{m} \geq \dim(R)$. The equality is from Nakayama, the inequality from PIT.

Definition 3.3. *Let X be a variety and $x \in X$.*

1. *X is nonsingular at x if $\mathscr{O}_{X,x}$ is a regular local ring.*
2. *Otherwise, X is singular at x.*
3. *X is nonsingular if all points $x \in X$ are nonsingular.*

Exercise: S is a commutative ring, and $M \subseteq S$ is a maximal ideal, then set $R = S_M$ a local ring. Then unique maximal ideal $\mathfrak{m} = MS_M$. Show that $S/M \simeq R/\mathfrak{m}$ and $M/M^2 = \mathfrak{m}/\mathfrak{m}^2$.

Example: Let $C = V(f) \subseteq \mathbb{A}^2$ a curve. Let $P = (0,0) \in C$. So $f = ax+by+HOT$. And $k[C] = k[x,y]/(f)$. Let $M = I(\{P\}) = (\bar{x}, \bar{y}) = (x,y)/(f) \subseteq k[C]$.

$\mathfrak{m}_P/\mathfrak{m}_P^2 = M/M^2 = (x,y)/(x^2,xy,y^2,f) = (x,y)/(x^2,xy,y^2,ax+by)$. So $P \in C$ is nonsing iff $\dim_k(M/M^2) = \dim(C) = 1$. This is iff $ax + by \neq 0 \in k[x,y]$, note that $ax + by \neq 0$ implies that $V(f)$ looks like $V(ax + by)$ close to the point.

Exercise: Find all singular points of $V(y^2 - x^3 - x^2), V(y^2 - x^3)$.

$X \subset \mathbb{A}^n$ a closed affine variety. Then $I = I(X) = (f_1, \dots, f_t) \subseteq S = k[x_1, \dots, x_n]$. Idea: X is nonsing at $P \in X$ iff X has a tangent space at P.

Definition 3.4 (Jacobi Matrix)**.** *Let $J_P = \left[\frac{\partial f_i}{\partial x_j}(P)\right]$ be a $t \times n$ matrix, we call this the Jacobi matrix.*

Note: If $\vec{v} \in k^n$ then $J_p \cdot \vec{v} \in k^t$ is the partial derivative of $(f_1, \dots, f_t)$ at p in the direction $\vec{v}$.

$\ker(J_P) = \{\vec{v} \in k^n : J_p \cdot \vec{v} = \vec{0}\}$ is a candidate for a tangent space.

Lemma 3.1. *Let $P \in X \subseteq \mathbb{A}^n$, then $\operatorname{rank}(J_p) + \dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = n$.*

Proof. Set $M = I(\{P\}) \subseteq S$. Define $d : M \to k^n$ by $d(f) = (\frac{\partial f}{\partial x_1}(P), \dots, \frac{\partial f}{\partial x_n}(P))$. d is surjective as $d(x_i - p_i) = e_i$. Note: $f, g \in S$ that $d(fg) = f(P)d(g) + d(f)g(P)$. So $d(M^2) = 0$. Thus, $d : M/M^2 \to k^n$ is an isomorphism if it is injective. It is injective as the two vector spaces are of the same dimension and it is surjective.

$d(f_i) = i^{th}$ row of J_P and $d(\sum g_i f_i) = \sum g_i(P) d(f_i)$ so $d(I)$ =row span of J_P in k^n. Thus $d : I + M^2/M^2 \to$ row span of J_P is an isomorphism. Thus $\operatorname{rank}(J_p) + \dim(M/I + M^2) = \dim(M/M^2) = n$. Finally $\mathscr{O}_{X,P} = (S/I)_M$ and $\mathfrak{m}_P/\mathfrak{m}_P^2 = (M/I)/(M/I)^2 = M/I + M^2$. ☐

Theorem 3.2. *$P \in X \subseteq \mathbb{A}^n$, then* $\operatorname{rank}(J_P) \leq n - \dim(\mathscr{O}_{X,P})$ *and* $\operatorname{rank} J_P = n - \dim \mathscr{O}_{X,P} \iff P$ *nonsingular.*

Proof. $\operatorname{rank}(J_p) = n - \dim(\mathfrak{m}_P/\mathfrak{m}_P^2) \leq n - \dim \mathscr{O}_{X,P}$.

We have equality iff $\dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = \dim \mathscr{O}_{X,P}$. □

Example: $X = V(z^2 - x^2y^2) \subseteq \mathbb{A}^3$ where $\operatorname{char}(k) \neq 2, 3$. Then $J = [-2xy^2, -2x^2y, 3z^2]$. $p \in X$ is a nonsingular point iff $\operatorname{rank}(J_p) = 3 - x = 1$, so $X_{sing} = V(z^3 - x^2y^2, xy^2, x^2y, z^2) = V(xy, z)$.

Exercise: X is a variety and $p \in X$. Then $\mathscr{O}_{X,P}$ is a domain iff p is in only one component.

Theorem 3.3. *Any Nötherian regular local ring is a domain. (in fact, a UFD, and even Macaulay)*

Conclude: The points on an intersection of two components are singular.

Proposition 3.4. *$X_{sing} \subseteq X$ is a closed subset of X.*

Proof. Let $X = X_1 \cup \ldots \cup X_m$ be the components of X. Then $X_{sing} = \bigcup_{i=1}^m (X_i)_{sing} \cup \bigcup_{i \neq j} X_i \cap X_j$. The latter are closed, so without loss of generality, X irreducible and affine.

$X \subseteq \mathbb{A}^n$ closed, $I(X) = (f_1, \ldots, f_t) \subseteq k[x_1, \ldots, x_n]$, $P \in X_{sing}$ iff $p \in X$ and $\operatorname{rank}(J_p) < n - \dim \mathscr{O}_{X,p} = n - \dim(X)$.

Let $m_1, \ldots, m_N$ be all of the minors of size $n - \dim(X)$ in $J = [\frac{\partial f_i}{\partial x_j}]$. $X_{sing} = X \cap V(m_1, \ldots, m_N)$. □

Fact: $X_{sing} \neq X$.

Lemma 3.5. *If $p \in X = V(g_1, \ldots, g_r) \subseteq \mathbb{A}^n$ and if* $\operatorname{rank} J_p(g_1, \ldots, f_r) = r$ *then $\mathscr{O}_{X,p}$ is regular local of dimension $n - r$.*

Proof. PIT implies that $\dim \mathscr{O}_{X,p} \geq n - r$.

$I(X) = (f_1, \ldots, f_t) \supseteq (g_1, \ldots, g_r)$. Thus, row span $J_p(f_1, \ldots, f_t) \supseteq$ row span $J_p(g_1, \ldots, g_r)$, so $r = \operatorname{rank} J_p(g_1, \ldots, g_r) \leq \operatorname{rank} J_p(f_1, \ldots, f_t) \leq n - \dim \mathscr{O}_{X,p} \leq r$. □

Theorem 3.6 (Implicit Function Theorem)**.** *If $f_1, \ldots, f_c$ are holomorphic functions in a classical nbhd of $p \in \mathbb{C}^n$. Suppose* $\det \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1 \leq i,j \leq c} \neq 0$. *Then $\exists$ holomorphic functions $w_1, \ldots, w_c$ on classical open subset of $\mathbb{C}^{n-c}$ and classical open subset $V \subseteq \mathbb{C}^n$ such that $p \in V$ and so that for all $z \in V$, $f_1(z) = \ldots = f_c(z) = 0$ iff $z - i = w_i(z_{c+1}, \ldots, z_n)$ for all $1 \leq i \leq c$.*

Theorem 3.7. *$X \subseteq \mathbb{C}^n$ a complex affine variety. $p \in X$ a nonsingular point. Then a classical neighborhood of p in X is holomorphic to a classical open subset of $\mathbb{C}^d$ where $d = \dim \mathscr{O}_{X,p}$.*

Proof. WLOG, X is irreducible. $I(X) = (f_1, \ldots, f_t)$ and $\operatorname{rank} J_p(f_1, \ldots, f_t) = n - d = c$. WLOG, $\det(\frac{\partial f_i}{\partial x_j}(p)) \neq 0$. Set $Y = V(f_1, \ldots, f_c) \subseteq \mathbb{C}^n$, then $\operatorname{rank} J_p(f_1, \ldots, f_c) = c$ implies that p is a nonsingular point of Y. So $\mathscr{O}_{Y,p}$ is regular local of dimension d.

Now, only one component of Y contains p, $p \in X$ and $X \subseteq Y$. $\dim X$ is the same as the dimension of the component of Y containing p, and as X is irreducible, X is the component of Y containing p. Then there exists open $U \subseteq \mathbb{A}^n$ such that $X \cap U = Y \cap U = V(f_1, \ldots, f_c) \cap U$. We apply the IFT, let $V \subset U$, $p \in V$, and $w_1, \ldots, w_c$ be as in the IFT.

Define $\pi : \mathbb{C}^n \to \mathbb{C}^d$ by $\pi(z) = (z_{c+1}, \ldots, z_n)$. Then $\pi : X \cap V \to \pi(X \cap V) \subseteq \mathbb{C}^d$ is an holomorphism, as we can get an inverse map $\pi^{-1} : \pi(X \cap V) \to X \cap V$ by $(z_{c+1}, \ldots, z_n) \mapsto (w_1(z), \ldots, w_c(z), z_{c+1}, \ldots, z_n)$. □

Corollary 3.8. *Every nonsingular complex variety variety is a complex manifold.*

If X is an affine variety, recall that pts in X are in 1-1 correspondence with max ideals in $k[X]$.

So X an irreducible variety, $p \in X$ corresponds to local rings $\mathscr{O}_{X,P} = \{f \in k(X) : f(P) \text{ defined }\}$.

Lemma 3.9. *X an irred var, $x, y \in X$, if $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{X,y}$ then $x = y$.*

Proof. Take open affine $x \in U \subseteq X$, $y \in V \subseteq X$. X is separated, so $U \cap V$ is affine and $k[X] \otimes k[Y] \to k[U \cap V]$ is surjective, and $k[U] \subseteq \mathscr{O}_{X,x} \subseteq \mathscr{O}_{X,y}$ with $k[V] \subseteq \mathscr{O}_{X,y}$, thus, $k[U \cap V] \subseteq \mathscr{O}_{X,y}$.

$k[V] \subseteq k[U \cap V] \subseteq \mathscr{O}_{X,y}$ proves that $y \in U \cap V$. Then $k[U \cap V] \cap \mathfrak{m}_y$ is a max ideal in $k[U \cap V]$ so it corresponds to a point in $U \cap V$ which maps to y under $U \cap V \to V$ the inclusion.

Thus, $x, y \in U \subseteq X$. U is affine. If $x \neq y$, then there is an $f \in k[U]$ such that $f(x) \neq 0$, $f(y) = 0$. Then $\frac{1}{f} \in \mathscr{O}_{X,x}$ and $\frac{1}{f} \notin \mathscr{O}_{X,y}$. □

Nonsingular Curves

Definition 3.5 (Curves)**.** *A curve is an irreducible variety of dimension one.*

Example: $C = \mathbb{A}^1$. $k[C] = k[t]$ and $k(C) = k(t)$. Let $0 \neq f \in k(t)$. What is the order of vanishing of f at 0?

$f = p/q$, $p, q \in k[t]$, we can write $p = t^n p_0$ and $q = t^m q_0$ with $p_0(0) \neq 0$ and $q_0(0) \neq 0$. $f = (p_0/q_0)t^{n-m}$, so $v(f) = n - m$ is the order of vanishing.

Note that $\mathscr{O}_{C,0} = k[t]_{(t)}$ with $m_0 = (t) \subseteq \mathscr{O}_{C,0}$ is a maximal ideal. Then $f = ut^{n-m}$ where $u = p_0/q_0$ is a unit in $\mathscr{O}_{C,0}$.

Definition 3.6 (Discrete Valuation Ring)**.** *A discrete valuation ring, or DVR, is a Nötherian regular local ring of dimension 1.*

Examples are $\mathscr{O}_{C,P}$ where C is a curve and P a nonsingular point.

Let $(R, \mathfrak{m})$ be a DVR, $\mathfrak{m} = (t)$, t is a uniformizing parameter, and $K = R_0$ is the field of fractions of R.

Claim: Any $f \in K^* = K \setminus \{0\}$ can be written $f = ut^n$ where $u \in R \setminus \mathfrak{m}$ a unit in R and $n \in \mathbb{Z}$. WLOG, $f \in R \setminus \{0\}$.

Assume that the claim is false, then choose a counterexample such that (f) is maximal. f is not a unit implies that $f \in \mathfrak{m}$, so $f = gt$ for some $g \in R$.

$(f) \subsetneq (g)$ as $(f) = (g) \Rightarrow g = hf \Rightarrow f = thf \Rightarrow th = 1$, contradiction.

So $g = ut^n$, $u \in R$ a unit implies that $f = ut^{n+1}$.

Check that if $f = ut^n$, u a unit, then u, n are unique. $n = \min\{p \in \mathbb{Z} : f \in (t^p) \subseteq K\}$.

Definition 3.7 (Valuation Map)**.** *$v : K^* \to \mathbb{Z}$ is a valuation map $v(f) = n$ if $f = ut^n$ with $u \in R \setminus \mathfrak{m}$.*

Note that $R = \{f \in K^* : v(f) \geq 0\} \cup \{0\}$ and $\mathfrak{m} = \{f \in K^* : v(f) > 0\} \cup \{0\}$.

Rules: $v(fg) = v(f) + v(g)$. $v(f + g) \geq \min(v(f), v(g))$ if $f, g, f + g \in K^*$. If $f = ut^n, g = vt^m$ and $n \leq m$, then $f + g = (u + vt^{m-n})t^n = u't^rt^n$.

Example: C a curve, $p \in C$ nonsing, then $R = \mathscr{O}_{C,p}$ is a DVR with $K = R_0 = k(C)$. $v_p : K(C)^* \to \mathbb{Z}$ is a valuation, $f \in k(C)^*$ $v_p(f)$ =the order of vanishing of f at p. $v_p(f) > 0$ iff $f \in \mathfrak{m}_p$, $v_p(f) = 0$ iff $f(p) \neq 0$ and $v_p(f) < 0$ iff f is not defined at p.

Lemma 3.10. *Let R be a DVR, $R_0 = K$ and S any ring such that $R \subseteq S \subseteq K$. Then $S = R$ or $S = K$.*

Proof. If $S \neq R$, then take $f \in S$, $f \notin R$. $\mathfrak{m} = (t)$, $f = ut^n$, $n < 0$, $S \supseteq R[f] = K$. □

Recall: A domain R is called integrally closed iff for all $f \in R_0$, f is integral over R implies that $f \in R$.

Theorem 3.11. *If R is any Nötherian Local Domain of dimension one, then R is regular iff R is integrally closed.*

Exercise: Prove $\Rightarrow$ (easy, as $DVR \Rightarrow PID \Rightarrow UFD \Rightarrow$ int closed, so prove the last one)

Proposition 3.12. *Let A be a domain. A is integrally closed iff A_P is integrally closed for all maximal ideals $P \subseteq A$.*

Proof. $\Rightarrow$: Easy

$\Leftarrow$: $A = \cap_{P \subseteq A} A_P$ over maximal ideals P. □

Note: A a f.g. domain over k, $X = \text{Spec}-m(A)$, then $A = k[X] = \cap_{p \in X} \mathscr{O}_{X,p} = \cap A_p$.

Definition 3.8 (Dedekind Domain)**.** *A Dedekind Domain is an integrally closed Nötherian domain of dimension 1.*

Note: If X is an irred affine variety, X nonsingular curve iff $k[X]$ is a Dedekind domain.

$\mathscr{O}_{X,p}$ a DVR for all $p \in X$ iff $k[X]_p$ integrally closed for all max ideals p by the theorem, each of these is integrally closed iff $k[X]$ is, and that is the def of a Dedekind domain.

Finiteness of Integral Closure

Let R be a finitely generated domain over k, $K = R_0$, and $K \subseteq L$ a finite field extension. Then $\overline{R}$ is defined to be the integral closure of R in L. That is, $\{f \in L : f \text{ integral over } R\}$. Fact $\overline{R}$ is an integrally closed domain with field of fractions L.

Let $f \in L$. Then $f^n + a_1 f^{n-1} + \ldots + a_n = 0$ with $a_i \in K$. Take $b \in R$ such that $ba_i \in R$ for all i. Then $(bf)^n + ba_1(bf)^{n-1} + \ldots + b^n a_n = 0$. Thus $bf \in \overline{R}$. $b \in R$, so $f \in (\overline{R})_0$.

Theorem 3.13 (Finiteness of Integral Closure)**.** *$\overline{R}$ is a finitely generated R-module.*

In particular: $\overline{R}$ is a finitely generated K algebra.

Definition 3.9 (DVR of K/k)**.** *$k \subseteq K$ is a field extension, a DVR of K/k is a subring $R \subseteq K$ such that:*

1. *R is a DVR*
2. *$R_0 = K$*
3. *$k \subseteq R$.*

Let K be a function field of dimension 1 over k.

i.e., $k \subseteq K$ is finitely generated as a field extension and it has transcendence degree 1.

Q: $\exists$ a nonsingular curve C such that $k(C) = K$?

Key construction: Let $f \in K \setminus k$. Then $k(f) \subseteq K$ is a finite field extension. It is finitely generated by assumption, and $\{f\}$ must be a transcendence basis for K/k, thus it is algebraic, and so it is a finite field extension.

Set $B = \overline{k[f]} \subseteq K$.

Finiteness of integral closure says that B is a finitely generated k-algebra, and so B is a Dedekind domain with $B_0 = K$.

Proposition 3.14. *$X = \operatorname{Spec} - m(B)$ is a nonsingular curve.*

1. *Points in X are in 1-1 correspondence with DVRs R of K/k such that $f \in R$.*
2. *Points in $V(f)$ correspond to the DVRs R of K/k such that $f \in \mathfrak{m}_R$.*

Proof. $X \to \{\text{DVRs } R \ni f\}$ is well-defined and injective.

Let R be a DVR of K/k with $f \in R$.

$k[f] \subseteq R \Rightarrow B \subseteq R$.

Set $M = B \cap \mathfrak{m}_R \subseteq B$ a prime ideal. Then $B_M \subseteq R \neq K \Rightarrow M \neq 0$ and so B_M is a DVR of K/k.

And so, the lemma implies that $B_M = R$.

Now we prove (b). $M \in V(f) \iff f \in M \iff f \in MB_M = \mathfrak{m}$. □

Corollary 3.15. *Every DVR R of K/k is the local ring of a nonsingular curve at some point. In particular, $R/\mathfrak{m}_R = K$.*

Proof. Let $f \in R \setminus k$, $B = \overline{k[f]} \subseteq K$.

Then $R = B_M$ for some maximal ideal $M \in \text{Spec} -m(B)$. So $R = \mathscr{O}_{X,p}$. □

Corollary 3.16. *Given $f \in K^*$, there are only finitely many DVRs R of K/k such that $f \in \mathfrak{m}_R$.*

Also have finitely many R such that $f \notin R$.

Proof. WLOG, $f \notin k$. Set $B = \overline{k[f]} \subseteq K$. $\{R : f \in \mathfrak{m}_R\}$ is in correspondence with $V(f) \subseteq \text{Spec} -m(B)$. PIT implies that $\dim V(f) = 0$. Thus, $V(f)$ is a finite set.

Note: $f \notin R$ iff $1/f \in \mathfrak{m}_R$. □

Definition 3.10. $C_K = \{$*DVRs of* $K/k\}$

Elements of C_K will be called "points" P. DVR given by P is R_P with maximal ideal $\mathfrak{m}_P$.

We define a topology on C_K to have as closed sets the finite sets and all of C_K. We also let $f \in K$ and $P \in C_K$, assume $f \in R_P$.

Definition 3.11. *$f(P)$ is defined to be the image of f by $R_P \to R_P/\mathfrak{m}_P = k$. i.e. $f(P) \cong f \mod \mathfrak{m}_P$.*

The regular functions on a nonempty open subset $U \subseteq C_K$ are then the set $k[U] = \cap_{p \in U} R_P \subseteq K$.

This makes C_K a SWF.

Note: If $U \subseteq C_K$ open, $f \in k[U]$ then $D(f) = \{P \in U : f(P) \neq 0\} = \{P \in U : f \notin \mathfrak{m}_P\}$ is open by the second corollary.

Example Let $K = k(t)$. The DVRs of K/k are $k[t]_{(t-a)}$ for $a \in k$ and $k[1/t]_{(1/t)}$.

Then C_K and $\mathbb{P}^1$ are in 1-1 correspondence, with $k[t]_{(t-a)}$ corresponding to $a \in \mathbb{A}^1$ and the other point corresponding to the point at infinity.

Note: If $f \in k[t]_{(t-a)}$ then $f(t) \cong f(a) \mod (t-a)$, $f(k[t]_{(t-a)}) = f(a)$.

Theorem 3.17. *C_K is a non-singular curve and $k(C_K) = K$.*

Proof. Take any $f \in K \setminus k$. $B = \overline{k[f]} \subseteq K$. $U = \{p \in C_K : f \in R_P\} \subseteq C_K$ is open.

We define $\phi : \text{Spec} -m(B) \to U$ by $M \mapsto B_M$. The prop implies that this is bijective.

ϕ is a homeomorphism, as the closed sets are the finite sets in both. It remains to show that this is a morphism of spaces with functions.

For $V \subseteq \text{Spec} -m(B)$ open, then $k[V] = \cap_{M \in V} B_M = \cap_{P \in \phi(V)} R_P = k[\phi(V)]$.

Thus, $\phi : \text{Spec} -m(B) \to U$ is an isomorphism of spaces with functions.

Note: If $P \in C_K$ then $f \in R_P$ or $1/f \in R_P$. Thus, $C_k = \text{Spec} -m(\overline{k[f]}) \cup \text{Spec} -m(\overline{k[f^{-1}]})$. This is an open affine cover, and so C_k is a prevariety.

Let $P, Q \in C_K$. Enough to find an open affine $U \subseteq C_K$ such that $P, Q \in U$. Take $f \in R_P \setminus \mathfrak{m}_P$ and $f \notin k$. ($f = 1 + t$, $(t) = \mathfrak{m}_P$).

If $f \in R_Q$ then P, Q are both in Spec $-m(\overline{k[f]})$.

Otherwise, $1/f \in R_Q$, so both are in Spec $-m(\overline{k[1/f]})$.

Thus C_K is a variety. It is nonsingular and of dimension one by construction. We must show that it is irreducible.

C_K is, in fact, irreducible because its proper closed sets are finite and C_K is infinite. □

Proposition 3.18. *Let C be an irreducible curve and $P \in C$ a nonsingular point. Let Y be any projective variety and $\phi : C \setminus \{P\} \to Y$ is any morphism of varieties. Then $\exists!$ extension $\phi : C \to Y$.*

Note: No points in Y are "missing".

Proof. $Y \subseteq \mathbb{P}^n$ closed subset. It is enough to make $\phi : C \to \mathbb{P}^n$.

WLOG $\phi(C \setminus \{p\}) \not\subseteq V_+(x_i)$ for all i. Set $U = D(x_0, x_1, \ldots, x_n) \subseteq \mathbb{P}^n$. $\phi(C \setminus \{p\}) \cap U \neq \emptyset$.

Set $f_{ij} = x_i/x_j \circ \phi \in k(C)$. Defined on $\phi^{-1}(U) \neq \emptyset$.

$v_P : k(C)^* \to \mathbb{Z}$ is the valuation given by $\mathscr{O}_{C,P}$.

Set $r_i = v_P(f_{i0})$ for $0 \leq i \leq n$. Choose j such that r_j minimal.

As $\frac{x_i}{x_j} = \frac{x_i/x_0}{x_j/x_0}$, $f_{ij} = f_{i0}/f_{j0}$, so $v_P(f_{ij}) = r_i - r_j \geq 0$. Thus, $f_{ij} \in \mathscr{O}_{C,P}$ for all i. Note that if $Q \in \phi^{-1}(U)$, then $\phi(Q) = (f_{0j}(Q) : f_{1j}(Q) : \ldots : f_{jj}(Q) = 1 : \ldots : f_{nj}(Q))$.

We can then extend ϕ to P by this expression. Then ϕ is a morphism on $\phi^{-1}(U) \cup \{p\}$ and so $\phi : C \to \mathbb{P}^n$ is a morphism. □

Lemma 3.19. *$R \subseteq K$ is a local ring, $k \subseteq R$, R is not a field. Then R is contained in some discrete valuation ring of K/k.*

Proof. Set $B = \overline{R} \subseteq K$. Lying over implies that there exists some maximal ideal $M \subseteq B$ such that $M \cap R = \mathfrak{m}_R$.

Claim: B_M is a DVR of K/k.

Let $0 \neq f \in \mathfrak{m}_R$. $S = \overline{k[f]}$ is a Dedekind domain, $S \subseteq B$. $\tilde{M} = M \cap S$ is a maximal ideal of S.

Thus $S_{\tilde{M}}$ is a DVR of K/k. $S_{\tilde{M}} \subseteq B_M \subsetneq K$ and a lemma from before says that we have equality of $S_{\tilde{M}}$ and B_M. □

Theorem 3.20. *C_K is a projective curve.*

Proof. Let $f \in K \setminus k$. $U = \text{Spec}\,-m(\overline{k[f]})$, $V = \text{Spec}\,-m(\overline{k[f^{-1}]})$, and $C_K = U \cup V$ an open affine cover.

$U \subseteq \mathbb{A}^N$ closed. $\overline{U} \subseteq \mathbb{P}^N$ projective closure.

The proposition implies that the inclusion $U \to \overline{U}$ extends to a morphism $\varphi_1 : C_K \to \overline{U}$.

Similarly, we take $\overline{V}$ to be the projective closure of V. Then $V \to \overline{V}$ extends to $\varphi_2 : C_K \to \overline{V}$.

We now define $\varphi : C_k \to \overline{U} \times \overline{V}$ by $\varphi(P) = (\varphi_1(P), \varphi_2(P))$. Set $Y = \overline{\varphi(C_K)} \subseteq \overline{U} \times \overline{V}$.

Y is a projective variety.

Claim: $\varphi : C_K \to Y$ is an isomorphism.

Note: $\varphi(U) \subseteq U \times \overline{V}$ is a closed subset. Let $\psi = \varphi_2 \times \mathrm{id} : U \times \overline{V} \to \overline{V} \times \overline{V}$. $\varphi(U) = \{(u, v) \in U \times \overline{V} : \varphi_2(u) = v\} = \psi^{-1}(\Delta_{\overline{V}})$ closed. Thus, $\varphi(U) = Y \cap (U \times \overline{V})$, which implies that $\varphi : U \to Y \cap (U \times \overline{V})$ is bijective, isomorphism.

So $\pi_U : Y \cap (U \times \overline{V}) \to U$ is the inverse.

Similarly, $\varphi : V \to Y \cap (\overline{U} \times V)$ is an isomorphism.

Note: $k(Y) = k(C_K) = K$, and forall $P \in C_K$, $\mathscr{O}_{Y,\varphi(P)} = R_P \subseteq K$. Thus, φ is injective.

For surjective, let $y \in Y$. Then $k \subseteq \mathscr{O}_{Y,y} \subseteq K$ is a local ring. By the lemma, $\mathscr{O}_{Y,y} \subseteq R_P$ for some $P \in C_K$. $\mathscr{O}_{Y,y} \subseteq R_P = \mathscr{O}_{Y,\varphi(P)}$ so $y = \varphi(P)$. □

Corollary 3.21. *Any curve is birational to some nonsingular projective curve.*

Corollary 3.22. *X is any nonsingular curve, then $X \cong$ some open subset of C_K, $K = k(X)$.*

Proof. $\varphi : X \to C_K$ by $\varphi(x) = P$ where $P \in C_K$ such that $R_P = \mathscr{O}_{X,x} \subseteq K$.

Injectivity is clear. Claim: $\varphi(X) \subseteq C_K$ is open. Take $U \subseteq X$ open affine. $k[U]$ is generated by $f_1, \dots, f_n$. $P \in \varphi(U)$ iff $k[U] \subseteq R_P$, iff $f_i \in R_P$ for all i.

Thus, $\varphi(U) = \cap_{i=1}^{n} \operatorname{Spec} - m(\overline{k[f_i]})$ open. Thus $\varphi(X) \subseteq C_K$ is open.

$\varphi : X \to \varphi(X)$ is a homeomorphism. To check that φ is an isomorphism, then $U \subseteq X$ open gives $k[U] \cap_{x \in U} \mathscr{O}_{X,x} = \cap_{P \in \varphi(U)} R_{\varphi(x)} = k[\varphi(U)]$. □

Exercise: Two nonsingular projective curves are isomorphic iff they have the same function field.

Degree of Projective Varieties in $\mathbb{P}^n$

Bezout: $f_1, \dots, f_n \in k[x_0, \dots, x_n]$ homogeneous of degrees $d_1, \dots, d_n$. Then $V_+(f_1, \dots, f_n)$ has cardinality at most $d_1 \dots d_n$ or is infinite. If it is finite and counted with multiplicity, then it is equal to $d_1 \dots d_n$.

Classical Definition: $X \subseteq \mathbb{P}^n$ closed, then $\deg(X) = \#(X \cap V)$ where $V \subseteq \mathbb{P}^n$ is a linear subspace with $\dim V + \dim X = n$.

e.g. $f \in S = k[x_0, \dots, x_n]$ a square-free homogeneous polynomial. Then $\#(V_+(f) \cap \text{ general line}) = \deg f$.

Warning: $V_+(xz - y^2)$, $V_+(x) \subseteq \mathbb{P}^2$. These are isomorphic but have different degrees. So degree is not a property of a projective variety, but rather one of the embedding into projective space.

Example: $f \in S$ is square-free, $\deg f = d$. Then $X = V_+(f)$, $I(X) = (f)$, $R = S/(f)$ is the projective coordinate ring.

Note: $\dim_k(S_m) = \binom{m+n}{n}$, where S_m is all forms of degree m. This is $\frac{1}{n!}(m+n)(m+n-1)\dots(m+1)$, which is actually a polynomial in m of degree n with lead coefficient $\frac{1}{n!}$.

Consider $0 \to S \xrightarrow{f} S \to R \to 0$ implies that $0 \to S_{m-d} \to S_m \to R_m \to 0$ is exact. Then $\dim R_m = \dim S_m - \dim S_{m-d}$, which is $\binom{m+n}{n} - \binom{m+n-d}{n}$, which is $\frac{1}{n!}(m+n)\dots(m+1) - \frac{1}{n!}(m+n-d)\dots(m+1-d)$. Which is a polynomial of degree $n-1$.

This has lead coefficient $\frac{1}{n!}\left(\sum i - \sum(i-d)\right) = \frac{nd}{n!} = \frac{d}{(n-1)!}$.

Recall: A graded S-module is a module M with a decomposition $M = \oplus_{d \in \mathbb{Z}} M_d$ as abelian group such that $S_m M_d \subset M_{m+d}$.

Definition 3.12. $\mathrm{Ann}(M) = \{f \in S : fM = 0\}$ *is a homogeneous ideal,* $\mathrm{Supp}(M) = V_+(\mathrm{Ann}(M)) \subseteq \mathbb{P}^n$.

Reason for Supp is $x \in \mathbb{P}^n$, $P = I(\{x\}) \subseteq S$ is a homogeneous prime ideal. $M_P \neq 0$ iff $P \supseteq \mathrm{Ann}(M)$, iff $x \in \mathrm{Supp}(M)$.

Example: $X \subseteq \mathbb{P}^n$ closed, then $\mathrm{Ann}(S/I(X)) = I(X)$, so $\mathrm{Supp}(S/I(X)) = V_+(I(X)) = X$.

Note: M_d is a k-vector space for all $d \in \mathbb{Z}$ because $S_0 M_d \subseteq M_d$. If M is a finitely generated graded S-module, then $\dim_k M_d < \infty$ forall d.

Definition 3.13 (Hilbert Function)**.** $\mathscr{H}_M(d) = \dim_k M_d$ *is the Hilbert function of M.*

Theorem 3.23. *If M is a finitely generated graded S-module then $\exists! P_M(z) \in \mathbb{Q}[z]$ such that $P_M(d) = \dim_k(M_d)$ for all d sufficiently large. We call $P_M(z)$ the Hilbert Polynomial.*

Definition 3.14 (Numerical Polynomial)**.** *$P(x) \in \mathbb{Q}[z]$ is a numerical polynomial if $P(d) \in \mathbb{Z}$ for all d sufficiently large in $\mathbb{Z}$.*

Example: $\binom{z}{m} = \frac{1}{m!} z(z-1)\dots(z-m+1)$.

Note: $\left\{\binom{z}{m} : m \in \mathbb{N}\right\}$ is a basis over $\mathbb{Q}$ for $\mathbb{Q}[z]$.

Lemma 3.24. $P(z) = c_0 + c_1\binom{z}{1} + \dots + c_r\binom{z}{r} \in \mathbb{Q}[z]$, $c_i \in \mathbb{Q}$. *Then TFAE*

1. *$P(z)$ is a numerical polynomial.*
2. *$P(d) \in \mathbb{Z}$ for all $d \in \mathbb{Z}$.*
3. *$c_i \in \mathbb{Z}$.*

Proof. $3 \Rightarrow 2 \Rightarrow 1$ are easy, so we need $1 \Rightarrow 3$.

We know that $\binom{z+1}{m} - \binom{z}{m} = \binom{z}{m-1}$. Thus $P(z+1) - P(z) = c_1 + c_2\binom{z}{1} + \dots + c_r\binom{z}{r-1}$.

We perform induction on r: $P(z)$ is numeric, then $P(z+1) - P(z)$ is also numeric. Thus, $c_1, \dots, c_r \in \mathbb{Z}$, and so c_0 must also be an integer. □

Theorem 3.25 (Affine Dimension Theorem)**.** *$X, Y \subseteq \mathbb{A}^n$ closed and irreducible, then $Z \subseteq X \cap Y$ component has $\dim Z \geq \dim X + \dim Y - n$.*

Proof. $X \cap Y = (X \times Y) \cap \Delta_{\mathbb{A}^n} = (X \times Y) \cap V(x_1 - y_1, \ldots, x_n - y_n) \subseteq \mathbb{A}^n \times \mathbb{A}^n$, use PIT. □

WARNING: Does not prevent $X \cap Y = \emptyset$.

Theorem 3.26 (Projective Dimension Theorem)**.** *$X, Y \subseteq \mathbb{P}^n$ closed irreducible, $Z \subseteq X \cap Y$ a component, then $\dim Z \geq \dim X + \dim Y - n$. If $\dim X + \dim Y - n$ is nonnegative, then $X \cap Y$ is nonempty.*

Proof. First statement follows from ADT.

Set $s = \dim X, t = \dim Y$. $C(X) = \overline{\pi^{-1}(X)} \subseteq \mathbb{A}^{n+1}$, then $\dim C(X) = s+1$, $\dim C(Y) = t + 1$.

Every component of $C(X) \cap C(Y)$ has dimension $\geq s + 1 + t + 1 - n - 1 = s + t - n + 1 \geq 1$, and $0 \in C(X) \cap C(Y)$, so such a component exists. □

Definition 3.15 (Twisted Module)**.** *Let $\ell \in \mathbb{Z}$, $M(\ell)$ is the twisted module given by $M(\ell)_d = M_{\ell+d}$. i.e., we are shifting the grading, but doing nothing else.*

Definition 3.16 (Homogeneous Homomorphism)**.** *A homomorphism $\varphi : M \to N$ of graded S-modules is homogeneous if $\varphi(M_d) \subseteq M_d$ for all d.*

Definition 3.17 (Homogeneous Submodule)**.** *A submodule $N \subseteq M$ is homogeneous if $N = \oplus_{d \in \mathbb{Z}}(N \cap M_d)$.*

This implies that N and $M/N = \oplus M_d/(N \cap M_d)$ are graded and $0 \to N \to M \to M/N \to 0$ is a short exact sequence of homogeneous homomorphisms.

Note: Let $m \in M$ be homogeneous of degree ℓ, that is, $m \in M_\ell$, then $I = \mathrm{Ann}(m) \subseteq S$ is a homogeneous ideal. Set $N = S \cdot m \subseteq M$, then $0 \to I \to S \to N \to 0$ is a short exact sequence, but it is not quite of homogeneous maps, unless we change S, I to $S(-\ell), I(-\ell)$. Thus, $(S/I)(-\ell)$ is isomorphic to $N = S \cdot m \subseteq M$.

<u>Exercise</u>: M f.g. module over a Nötherian ring, then M is a Nötherian module.

Lemma 3.27. *M a f.g. graded module over a Nötherian graded ring S, then $\exists$ filtration $0 = M^0 \subseteq \ldots \subseteq M^r = M$ such that $M^i \subseteq M$ is a homogeneous submodule and $M^i/M^{i-1} \simeq S/P_i(\ell_i)$ where P_i is a homogeneous prime ideal and $\ell_i \in \mathbb{Z}$.*

Proof. Let $N \subseteq M$ be maximal homogeneous submodule such that the lemma is true for N.

Claim: $N = M$. Else $M'' = M/N \neq 0$, Take $0 \neq m \in M''$ homogeneous such that $\mathrm{Ann}(m) \subseteq S$ is as large as possible.

Claim: $P = \mathrm{Ann}(m)$ prime ideal. $P \neq S$. Let $f, g \in S \setminus P$. Enough to show that $fg \notin P$. Note: $gm \neq 0$ and $\mathrm{Ann}(gm) \supsetneq \mathrm{Ann}(m)$, thus $\mathrm{Ann}(gm) = \mathrm{Ann}(m)$. So we must have $fg \notin \mathrm{Ann}(m)$, else $f \in \mathrm{Ann}(gm)$.

Thus, $Sm \simeq (S/P)(-\ell)$, $\ell = \deg m$. So $M \to M/N = M'' \supseteq Sm$, $\tilde{N} \subseteq M$ is the inverse image of Sm, and $N \subsetneq \tilde{N}$, and the lemma is true for $\tilde{N}$. □

Definition 3.18 (Eventually Polynomial)**.** *$f : \mathbb{Z} \to \mathbb{Z}$ is eventually polynomial if $\exists P(z) \in \mathbb{Q}[z]$ such that $f(n) = P(n)$ for all $n >> 0$.*

Set $\Delta f(n) = f(n+1) - f(n)$

Lemma 3.28. *f is eventually polynomial of degree r iff Δf is eventually polynomial of degree $r-1$.*

Proof. $\Rightarrow$: Obvious.

$\Leftarrow$: Assume $\Delta f(n) = Q(n)$ for all $n >> 0$ where $Q(z) = c_1 + c_2 \binom{z}{1} + \ldots + c_r \binom{z}{r-1}$. Set $P(z) = c_1 \binom{z}{1} + \ldots + c_r \binom{z}{r}$.

Then $\Delta P = Q$, so $\Delta(f - P)(n) = 0$ for all $n >> 0$. Thus $f(n) - P(n) = c_0$ a constant for $n >> 0$. □

Set $S = k[x_0, \ldots, x_n]$, M a f.g. graded S-module.

<u>Hilbert Function</u>: $H_M(d) = \dim_k(M_d)$, and $\operatorname{Supp}(M) = V_+(\operatorname{Ann}(M)) \subseteq \mathbb{P}^n$.

Note: If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of graded S-modules, then $\operatorname{Supp}(M) = \operatorname{Supp}(M') \cup \operatorname{Supp}(M'')$.

$\supseteq$: $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(M') \cap \operatorname{Ann}(M'')$.

$\subseteq$: Let $x \in \mathbb{P}^n$. If $x \notin \operatorname{Supp}(M') \cup \operatorname{Supp}(M'')$, then $\exists f \in \operatorname{Ann}(M')$ such that $f(x) \neq 0$ and $\exists g \in \operatorname{Ann}(M'')$ such that $g(x) \neq 0$, then $fg \in \operatorname{Ann}(M)$, but $fg(x) \neq 0$.

$\operatorname{Ann}(M') \operatorname{Ann}(M'') \subseteq \operatorname{Ann}(M) \subseteq \operatorname{Ann}(M') \cap \operatorname{Ann}(M'')$.

Theorem 3.29. *M f.g. graded module over $S = k[x_0, \ldots, x_n]$ then $H_M(d)$ is eventually equal to a polynomial $P_M(d)$ of degree $\dim(\operatorname{Supp}(M))$.*

Proof. $\exists$ filtration $0 = M^0 \subset \ldots \subset M^r = M$. such that $M^i/M^{i-1} \simeq (S/P_i)(\ell_i)$ where $P_i \subseteq S$ is a homogeneous prime.

Note: $H_M(d) = \sum_{i=1}^r H_{M^i/M^{i-1}}(d) = \sum_{i=1}^r H_{S/P_i}(d + \ell_i)$.

$\operatorname{Supp}(M) = \cup_{i=1}^r \operatorname{Supp}(M^i/M^{i-1}) = \cup_{i=1}^r V_+(P_i)$. WLOG, $M = S/P$, P a homogeneous prime. Induction of $V_+(P)$:

If $P = (x_0, \ldots, x_n)$, then the theorem is true if we take $\dim \emptyset = \deg 0 = -1$.

Otherwise, some $x_i \notin P$. Set $I = P + (x_i)$. Then $V_+(I) \subsetneq V_+(P)$, so $\dim V_+(I) = \dim V_+(P) - 1$ by the projective dimension theorem.

By induction, $H_{S/I}(d)$ is eventually polynomial and $\deg(P_{S/I}) = \dim V_+(P) - 1$

$0 \to (S/P)(-1) \to S/P \to S/I \to 0$, so $\Delta H_{S/P}(d-1) = H_{S/P}(d) - H_{S/P}(d-1) = H_{S/I}(d)$, so $H_{S/P}(d)$ is eventually polynomial of degree equal to $\dim V_+(P)$. □

Note: $P_M(z) = c_0 + c_1 \binom{z}{1} + \ldots + c_r \binom{z}{r} \in \mathbb{Q}[z]$ is a numeric polynomial, so $c_i \in \mathbb{Z}$.

$r = \dim \operatorname{Supp}(M)$, so $r!(\text{lead coef of } P_M(z)) \in \mathbb{Z} \geq 0$

Definition 3.19 (Hilbert Polynomial of a Variety). *Let $X \subseteq \mathbb{P}^n$ be a closed subvariety of dimension r, then $P_X(z) = P_{S/I(X)}(z)$ is the Hilbert Polynomial for X.*

We now define $\deg(X) = r!$(*lead coef of* $P_X(z)$).

Examples

1. $\deg V_+(f) = \deg f$
2. $V \subseteq \mathbb{P}^n$ linear subspace. WLOG, $V = V_+S(x_{r+1}, \dots, x_n)$. $S/I(V) \simeq k[x_0, \dots, x_r]$, so $P_{S/I(X)}(d) = \binom{r+d}{r}$, so lead coef is $1/r!$, so we get degree 1.

Proposition 3.30. *$X_1, X_2 \subseteq \mathbb{P}^n$ closed, $\dim X_1 = \dim X_2 = r$, no components in common, then $\deg(X_1 \cup X_2) = \deg X_1 + \deg X_2$*

Proof. $I_1 = I(X_1)$, $X_2 = I(X_2)$.

$I(X_1 \cup X_2) = I_1 \cap I_2$, so $0 \to S/(I_1 \cap I_2) \to S/I_1 \oplus S/I_2 \to S/(I_1 + I_2) \to 0$.

The first takes $f \mapsto (f, f)$ and the second takes $(f, g) \mapsto f - g$. They are injective and surjective, and so we see this this is a short exact sequence. Thus $P_{X_1}(d) + P_{X_2}(d) = P_{X_1 \cup X_2}(d) + P_{S/(I_1+I_2)}(d)$, so $\mathrm{Supp}(S/(I_1 + I_2)) = V_+(I_1 + I_2) = X_1 \cap X_2$. $\dim < r$, so $\mathrm{LC}(P_{X_1}) + \mathrm{LC}(P_{X_2}) = \mathrm{LC}(P_{X_1 \cup X_2})$. □

Corollary 3.31. *If $X \subseteq \mathbb{P}^n$ has dim zero, then $\deg(X) =$the number of points in X.*

Definition 3.20 (Simple Module). *If R is a ring and M an R-module, then M is simple if M has no nontrivial submodules and $M \neq 0$. This is equivalent to $M \simeq R/P$ where $P \subseteq R$ is a maximal ideal.*

Definition 3.21 (Decomposition Series). *A decomposition series for M is a filtration $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r = M$ such that M_i/M_{i-1} is simple for all i.*

Definition 3.22 (Artinian Module). *If there exists a decomposition series, then M is an Artinian module and we define* $\mathrm{length}(M) = r$.

Assume that R is Nötherian and that M is a finitely generated R-module. Then there exists a filtration $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r = M$. such that M_i/M_{i-1} is isomorphic to R/P_i where $P_i \subseteq R$ is maximal.

Note: $\mathrm{Ann}(M) \subseteq P_i$ for all i.

Lemma 3.32. *If $P \subseteq R$ is a minimal prime over $\mathrm{Ann}(M)$ then M_P is an Artinian R_P-module which has $\mathrm{length}_{R_P}(M_P) = |\{i : P_i = P\}|$ in the filtration of M.*

Proof. $0 = (M_0)_P \subseteq (M_1)_P \subseteq \dots \subseteq (M_r)_P = M_P$, and $(M_i)_P/(M_{i-1})_P = (M_i/M_{i-1})_P = (R/P_i)_P$.

If $P = P_i$, then we get R_P/PR_P, else we get 0. □

$X \subseteq \mathbb{P}^n$ is a closed subvariety, $S = k[x_0, \ldots, x_n]$, then $\dim_k(S/I(X))_d = P_X(d)$ for all $d >> 0$. $\dim(X) = \deg(P_X)$ and $\deg(X) = (\dim X)!\,\mathrm{LC}(P_X)$

Assume $X \subseteq \mathbb{P}^n$ has pure dimension r. That is, all components of X have dimension r.

$Y = V_+(f) \subseteq \mathbb{P}^n$ a hypersurface such that no component of X is contained in Y. Also assume that $f \in S$ is square-free.

Let $Z \subseteq X \cap Y$ be a component. Then $\dim Z = r - 1$. Set $M = S/(I(X) + (f))$. $\mathrm{Supp}(M) = X \cap Y$, do if $P = I(Z)$, then P is minimal prime over $\mathrm{Ann}(M)$.

Definition 3.23 (Intersection Multiplicity)**.** $I(X \cdot Y; Z) = \mathrm{length}_{S_P}(M_P)$

Theorem 3.33. $\deg(X)\deg(Y) = \sum_{Z \subseteq X \cap Y} I(X \cdot Y; Z)\deg(Z)$.

Proof. Set $d = \deg(X)$, $e = \deg(Y)$.

Then we have a short exact sequence $0 \to S/I(X) \xrightarrow{f} S/I(X) \to M \to 0$.

This gives us $0 \to (S/I(X))_{\ell - e} \to (S/(I(X)))_\ell \to M_\ell \to 0$ will be a short exact sequence for each ℓ, where this is breaking it into homogeneous parts.

Thus $P_M(\ell) = P_X(\ell) - P_X(\ell - e)$. Thus $\mathrm{LC}(P_M) = r \cdot e \cdot \mathrm{LC}(P_X) = red/r! = \frac{de}{(r-1)!}$.

Take filtration $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_t = M$, $M_i \subseteq M$ is homogeneous and M_i/M_{i-1} is isomorphic to S/P_i where $P_i \subseteq P$ is a homogeneous prime ideal.

So $P_M(\ell) = \sum_{i=1}^t P_{M_i/M_{i-1}}(\ell) = \sum_{Q \subseteq S \texttt{ hom prime}} P_{S/Q}(\ell)|\{i : Q = P_i\}|$, and from this we get $\sum_{Z \subseteq X \cap Y; component} |\{i : P_i = I(Z)\}| P_Z(\ell) + LOT$. So we have $\mathrm{LC}(P_M) = \frac{1}{(r-1)!}\sum_{Z \subseteq X \cap Y} I(X \cdot Y; Z)\deg(Z)$. □

Corollary 3.34 (Bezout's Theorem)**.** *Let $X, Y \subseteq \mathbb{P}^2$ be curves of degree d and e such that $X \cap Y$ is a finite set, then $\sum_{P \in X \cap Y} I(X \cdot Y; P) = de$.*

Exercise: $P \in X \cap Y \subseteq \mathbb{P}^2$ then $I(X \cdot Y; P) = 1 \iff P$ is a nonsingular point of both X and Y and X and Y have different tangent directions at P.

Bezout's Theorem for $\mathbb{P}^n$

Idea: If $X \subseteq \mathbb{P}^n$ is closed and irreducible and $Y \subseteq \mathbb{P}^n$ is a hypersurface, then $[X] \cdot [Y] = \sum_{Z \subseteq X \cap Y} I(X \cdot Y; Z) \cdot [Z]$.

Suppose that $Y_1, Y_2, \ldots, Y_n \subseteq \mathbb{P}^n$ are hypersurfaces such that their intersection is finite. Then $(\ldots([\mathbb{P}^n] \cdot [Y_1]) \cdot [Y_2] \ldots) \cdot [Y_n] = \sum_{i=1}^N c_i [P_i]$ where $Y_1 \cap \ldots \cap Y_n = \{P_1, \ldots, P_N\}$. Then $\sum c_i = \prod_{j=1}^n \deg(Y_j)$.

In fact: $(\ldots([\mathbb{P}^n] \cdot [Y_1]) \ldots) \cdot [Y_m] = \sum_{i=1}^{N_m} c_i^{(m)}[Z_i]$ where $Y_1 \cap \ldots \cap Y_m = Z_1 \cup \ldots \cup Z_{N_m}$ components, then $\prod_{j=1}^m \deg(Y_j) = \sum_{i=1}^{N_m} c_i^{(m)} \deg(Z_i)$.

Fact: $c_i, c_i^{(m)}$ do not depend on the order of multiplication.

Corollary 3.35. *$Y_1 \cap \ldots \cap Y_n$ finite implies that $|Y_1 \cap \ldots \cap Y_n| \leq \prod \deg(Y_j)$.*

Useful Fact: $X_1, \ldots, X_m \subseteq \mathbb{P}^n$ irreducible closed, $d \in \mathbb{N}$ then there exists irreducible hypersurface $Y \subseteq \mathbb{P}^n$ of degree d such that $X_i \not\subseteq Y$ for all i.

$v_d : \mathbb{P}^n \to \mathbb{P}^N$, $N = \binom{n+d}{n} - 1$ the veronese map, then $Y = \mathbb{P}^n \cap H$, $H \subseteq \mathbb{P}^N$ hyperplane.

Intrinsic: {hypersurfaces of degree d in $\mathbb{P}^n$} $\leftrightarrow$ $\mathbb{P}(S_d)-$nonsquarefree by $V_+(f) \leftrightarrow [f]$

{irreducible hypersurfaces} $\leftrightarrow \mathbb{P}(S_d) \backslash \cup_{p+q=d} \varphi_{p,q}(\mathbb{P}(S_p) \times \mathbb{P}(S_q))$ where $\varphi_{p,q} : \mathbb{P}(S_p) \times \mathbb{P}(S_q) \to \mathbb{P}(S_{p+q}) : [f] \times [g] \mapsto [fg]$.

$\binom{n+p}{n} + \binom{n+q}{n} \leq \binom{n+p+q}{n}$ so $U \subseteq \mathbb{P}(S_d)$ is a dense open subset.

Now, if $X \subseteq \mathbb{P}^n$ is closed, then $X \subseteq V_+(f)$ iff $f \in I(X)_d$. Thus

{hypersurfaces$\not\supseteq X$} $\leftrightarrow W_X = \mathbb{P}(S_d) \setminus \mathbb{P}(I(X)_d)$. Now, we can conclude that {irreducible hypersurfaces Y in $\mathbb{P}^n$ of degree d such that $X_i \not\subseteq Y$ for all i correspond to $U \cap W_{X_1} \cap \ldots \cap W_{X_m} \subseteq \mathbb{P}(S_d)$ is still a dense open subset.

4 Sheaves

Definition 4.1 (Presheaf)**.** *Let X be a topological space. A presheaf $\mathscr{F}$ of abelian groups on X is an assignment $U \mapsto \mathscr{F}(U)$ of an abelian group $\mathscr{F}(U)$ to each open subseteq U of X plus group homomorphisms $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ whenever $V \subseteq U \subseteq X$ open with*

1 $\rho_{UU} = \mathrm{id}$

2 $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ *when* $W \subseteq V \subseteq U$

Notation: elements of $\mathscr{F}(U)$ are called sections of $\mathscr{F}$ over U, ρ_{UV} are called restriction maps and $s \in \mathscr{F}(U)$, $V \subseteq U$, then $s|_V = \rho_{UV}(s)$. Also, say $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$.

Definition 4.2 (Sheaf)**.** *A sheaf is a presheaf $\mathscr{F}$ such that*

3 *If $s \in \mathscr{F}(U)$ and $U = \cup V_i$ an open cover, then $s|_{V_i} = 0$ iff $s = 0$.*

4 *If $U = \cup V_i$ an open cover and have sections $s_i \in \mathscr{F}(V_i)$ and $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i, j, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$.*

Note: The section s of axiom 4 is unique by axiom 3.

Remark: We can easily define sheaves of rings, sets, modules, etc.

Examples:

1. X is a SWF, then we can define $\mathscr{O}_X$ of k-algebras by $\mathscr{O}_X(U) = k[U]$. This sheaf is called the structure sheaf of X.

2. Let M be a manifold, $\mathscr{O}_M(U) = \{C^\infty \ f : U \to \mathbb{R}\}$. Then there is $\pi : TM \to M$ the tangent bundle, $\pi^{-1}(x) = T_x M$. Then $TM \leftrightarrow$ a sheaf $\mathscr{T}$ of $\mathscr{O}_M$-modules. $\mathscr{T}(U) = \{s : U \to TM$ such that $\pi(s(x)) = x$ for all $x \in U\}$. $\mathscr{T}(U)$ is a $\mathscr{O}_M(U)$-module.

Definition 4.3 (Morphism of Sheaves)**.** *A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of (pre)sheaves consists of homomorphisms $\varphi_U : \mathscr{F} \to \mathscr{G}$ for all $U \subseteq X$ open such that $s \in \mathscr{F}$, $V \subseteq U$ open, then $\varphi_U(s)|_V = \varphi_V(s|_V) \in \mathscr{G}(V)$ and*

$$\begin{array}{ccc} \mathscr{G}(V) & \longrightarrow & \mathscr{G}(U) \\ \downarrow{\scriptstyle \varphi_V} & & \downarrow{\scriptstyle \varphi_U} \\ \mathscr{F}(V) & \longrightarrow & \mathscr{F}(U) \end{array}$$

If $\varphi : \mathscr{F} \to \mathscr{G}$ and $\psi : \mathscr{G} \to \mathscr{H}$ are morphisms, we define $\psi \circ \varphi : \mathscr{F} \to \mathscr{H}$ by $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U$. An isomorphism is a morphism with an inverse morphism.

Definition 4.4 (The stalk of $\mathscr{F}$ at x)**.** *If $\mathscr{F}$ is a presheaf on X, $x \in X$ then the stalk of $\mathscr{F}$ at x is $\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}(U)$. This means*

1. *$\mathscr{F}_x$ is an abelian group (or whatever)*
2. *If $x \in U$ we have homomorphism $\rho_{Ux} : \mathscr{F}(U) \to \mathscr{F}_x$, if $x \in V \subseteq U$ then $\rho_{Ux} = \rho_{Vx} \circ \rho_{UV}$.*
3. *If G is any abelian group with homomorphisms $\Theta_U : \mathscr{F}(U) \to G$ such that $\Theta_U = \Theta_V \circ \rho_{UV}$ for all $x \in V \subseteq U$, then there exists a unique group homomorphism $\Theta_x : \mathscr{F}_x \to G$ such that $\Theta_U = \Theta_x \circ \rho_{Ux}$*

Example: X a variety, $x \in X$. Then $\mathscr{O}_{X,x} = \varinjlim_{U \ni x} \mathscr{O}_U$ =local ring of X at x.

Construction $\mathscr{F}_x = \left(\bigoplus_{U \ni x} \mathscr{F}(U)\right) / \langle (0, \dots, 0, s, 0, \dots, 0, -s|_V, 0, \dots, 0) : \forall s \in \mathscr{F}(U), x \in V \subseteq U \rangle$.

Notation: If $s \in \mathscr{F}_U$ and $x \in U$ write $s_x = \rho_{Ux}(s) \in \mathscr{F}_x$.

Exercise Let $\mathscr{F}$ be a presheaf.

1. All elements of $\mathscr{F}_x$ can be written as s_x for some $s \in \mathscr{F}(U)$, $x \in U$.
2. $s \in \mathscr{F}(U)$, $x \in U$, $s_x = 0 \in \mathscr{F}_x$ iff $s|_V = 0 \in \mathscr{F}(V)$ for $x \in V \subset U$.

Exercise: $\mathscr{F}$ is a sheaf. $s \in \mathscr{F}(U)$. $s = 0 \iff s_x = 0$ for all $x \in U$.

Note: A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ gives a homomorphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$, $\varphi_x(s_x) = \varphi_U(s)_x$, $s \in \mathscr{F}(U), x \in U$.

$$\begin{array}{ccc} \mathscr{G}_x & \xrightarrow{x \in U:} & \mathscr{G}(U) \\ \downarrow & & \downarrow \\ \mathscr{F}_x & \dashrightarrow & \mathscr{F}(U) \end{array}$$

Proposition 4.1. *$\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves. φ is an isomorphism iff $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ are isomorphisms for all $x \in X$.*

Proof. $\Rightarrow$: Clear

$\Leftarrow$: We must show that $\varphi : \mathscr{F}(U) \to \mathscr{G}(U)$ are isomorphisms. φ_U is injective as $s \in \mathscr{F}(U)$, if $\varphi_U(s) = 0 \in \mathscr{G}(U)$ then $\varphi_x(s_x) = \varphi_U(s)_x = 0$, but φ_x is injective, and so $s_x = 0$ for all $x \in U$, so $s = 0$.

To see that φ_U is surjective, take $t \in \mathscr{G}(U)$. φ_x surjective implies that $t_x = \varphi_x(s(x)_x) \in \mathscr{G}_x$ for some $s(x) \in \mathscr{F}(V_x)$ where $V_x \subseteq U$ are open subsets containing x.

Now $t_x = \varphi_{V_x}(s(x))_x$. We can make V_x smaller such that $t|_{V_x} = \varphi_{V_x}(s(x)) \in \mathscr{G}(V_x)$. $\varphi(s(x)|_{V_x \cap V_y}) = t|_{V_x \cap V_y} \varphi(s(y)|_{V_x \cap V_y})$. Thus $s(x)|_{V_x \cap V_y} = s(y)|_{V_x \cap V_y}$.

Patch: There exists a unique $s \in \mathscr{F}(U)$ such that $s|_{V_x} = s(x) \in \mathscr{F}(V_x)$ for all $x \in U$. $\varphi_U(s)|_{V_x} = \varphi_{V_x}(s|_{V_x}) = \varphi_{V_x}(s(x)) = t|_{V_x}$. Thus $\varphi_U(s) = t \in \mathscr{G}(U)$. □

Remark: $\mathscr{F}$ is a sheaf on X, $U \subseteq X$ is open. Define $\mathscr{F}|_U$ to be the sheaf on U by $\Gamma(V, \mathscr{F}|_U) = \Gamma(V, \mathscr{F})$

Example: Let $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Define sheaves $\mathscr{F}, \mathscr{G}$ by $\mathscr{F}(U) = \{f : U \to \mathbb{R} : f \text{ is locally constant}\}$ and $\mathscr{G}(U) = \{g : U \to \mathbb{R} : \frac{\partial g}{\partial \theta} = 1\}$. If $U \subsetneq S^1$ open then $\mathscr{F}|_U \simeq \mathscr{G}_U$ by $f(x,y) \mapsto f(x,y) + \mathrm{Arg}(x,y)$, so $\mathscr{F}_x \simeq \mathscr{F}_x$ for all $x \in S^1$ but $\mathscr{F}$ and $\mathscr{G}$ are not isomorphic as $\mathscr{F}(S^1) = \mathbb{R}$ and $\mathscr{G}(S^1) = \emptyset$.

Sheafification

Let $\mathscr{F}$ be a presheaf on X, define a sheaf $\mathscr{F}^+$ as follows: $U \subseteq X$ open set $\mathscr{F}^+(U)$ to be the set of all functions $s : U \to \coprod_{x \in U} \mathscr{F}_x$ such that

1. $s(x) \in \mathscr{F}_x$ for all x
2. $\forall x \in U$ there exists an open set V with $x \in V \subseteq U$ and $t \in \mathscr{F}(V)$ such that $s(y) = t_y \in \mathscr{F}_y$ forall $y \in V$.

Definition 4.5. *We define a morphism $\Theta : \mathscr{F} \to \mathscr{F}^+$ by $t \in \mathscr{F}(U)$, $\Theta_U(t) = [x \mapsto t_x] \in \mathscr{F}^+(U)$.*

Exercise: $\Theta_x : \mathscr{F}_x \to \mathscr{F}_x^+$.

Proposition 4.2. *Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf. $\varphi : \mathscr{F} \to \mathscr{G}$ is any morphism. Then there exists a unique $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}$ such that $\varphi = \varphi^+ \circ \Theta$.*

Proof. Let $s \in \mathscr{F}^+(U)$. i.e. $s : U \to \coprod_{x \in U} \mathscr{F}_x$. For $x \in U$ choose $t(x) \in \mathscr{F}(V_x)$ s.t. $x \in V_x \subset U$ open and $s(y) = t(x)_y$ for all $y \in V_x$.

Set $\tau(x) = \varphi_{V_x} = \varphi_{V_x}(t(x)) \in \mathscr{G}(V_x)$.

If $y \in V_x$, then $\tau(x)_y = \varphi(t(x))_y = \varphi_y(t(x)_y) = \varphi_y(s(y)) \in \mathscr{G}_y$. Thus, $\tau(x_1)|_{V_{x_1} \cap V_{x_2}} = \tau(x_2)|_{V_{x_1} \cap V_{x_2}}$, which gives that $\{\tau(x)\}$ glue to $\tau \in \mathscr{G}(U)$. Set $\varphi_U(s) = t$. Exercise: Check the details! □

Definition 4.6. *$\varphi : \mathscr{F}' \to \mathscr{F}$ is a morphism of sheaves.*

We say φ is injective if φ_U is injective for all $U \subseteq X$. If φ is injective, then we say $\mathscr{F}' \subseteq \mathscr{F}$ is a subsheaf as $\mathscr{F}'(U) \subseteq \mathscr{F}(U)$.

Exercise: φ injective iff $\varphi_x : \mathscr{F}'_x \to \mathscr{F}_x$ is injective for all $x \in X$.

Consequence: If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves such that $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ injective for all $U \subseteq X$ open, then $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ is injective.

Proof. Check: $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ injective for all $p \in X$. If $\varphi_p(s_p) = 0 \in \mathscr{G}_p$ for $s \in \mathscr{F}(U)$, $p \in U$ then $\varphi_U(s)_p = 0$ so $\varphi_U(s)|_V = 0$ for $p \in V \subseteq U$, so $\varphi_V(s|_V) = 0$. Thus $s|_V = 0$, so $s_P = 0$. □

In particular: If a presheaf $\mathscr{F}$ is a subpresheaf of a sheaf $\mathscr{G}$ then $\mathscr{F}^+$ is a subsheaf of $\mathscr{G}$.

Definition 4.7 (Kernel and Image)**.** *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups.*

Then $\ker\varphi =$ *the sheaf* $U \mapsto \ker(\varphi_U) \subseteq \mathscr{F}(U)$.

$\operatorname{Im}\varphi =$ *the sheafification of the presheaf* $U \mapsto \operatorname{Im}(\varphi_U) \subset \mathscr{G}(U)$.

Note, $\ker\varphi \subset \mathscr{F}$, $\operatorname{Im}\varphi \subset \mathscr{G}$ are subsheaves, and φ injective iff $\ker\varphi = 0$.

Definition 4.8 (Surjective)**.** *φ is surjective if* $\operatorname{Im}(\varphi) = \mathscr{G}$.

WARNING: φ surjective DOES NOT IMPLY that φ_U is surjective.

Exercise: φ surjective iff $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ surjective for all $p \in X$.

Definition 4.9 (Quotient Sheaf)**.** *$\mathscr{F}' \subset \mathscr{F}$ a subsheaf, then $\mathscr{F}/\mathscr{F}' =$ the sheafification of* $[U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)]$

We have a surjective morphism $\mathscr{F} \to \mathscr{F}/\mathscr{F}'$ which has kernel $\mathscr{F}'$.

Notation: A sequence of sheaves $\to \mathscr{F}^i \xrightarrow{\phi_i} \mathscr{F}^{i+1} \xrightarrow{\phi_{i+1}} \mathscr{F}^{i+2} \to \dots$ is a complex if $\phi_{i+1} \circ \phi_i = 0$ for all i and is exact if $\operatorname{Im}\phi_i = \ker\phi_{i+1}$ for all i.

Equivalently, complexes and exact sequences of the stalks in the category of abelian groups.

Example, $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is exact iff $\mathscr{F}' \subset \mathscr{F}$ and $\mathscr{F}'' \simeq \mathscr{F}/\mathscr{F}'$.

Definition 4.10. *$f : X \to Y$ a continuous map, $\mathscr{F}$ a sheaf on X, then $f_*\mathscr{F}$ is a sheaf on Y defined by $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$.*

Example: X a variety, $Y \subseteq X$ closed, $i : Y \to X$ the inclusion, $U \subseteq X$ open, then $\mathscr{I}_Y(U) = \{f \in \mathscr{O}_X(U) | f(y) = 0, \forall y \in U \cap Y\}$, $\mathscr{I}_Y \subseteq \mathscr{O}_X$ is a subsheaf of ideals. Then $\mathscr{O}_X(U)/\mathscr{I}_Y(U)$ are the regular functions $U \cap Y \to k$ which can be extended to all of U.

$\mathscr{O}_X/\mathscr{I}_Y \simeq i_*\mathscr{O}_Y$ because we can extend locally. We have exact sequence $0 \to \mathscr{I}_Y \to \mathscr{O}_X \to i_*\mathscr{O}_Y \to 0$, which will often be written $0 \to \mathscr{I}_Y \to \mathscr{O}_X \to \mathscr{O}_Y \to 0$.

Example, $f : X \to y$ a morphism of SWFs, we get morphism $f^* : \mathscr{O}_Y \to f_*\mathscr{O}_X$ by $\mathscr{O}_Y(V) \to f_*(\mathscr{O}_X(V)) = \mathscr{O}_X(f^{-1}(V))$, $h \mapsto h \circ f = f^*h$.

Exercise: Find a morphism $f : X \to Y$ of varieties such that $f^* : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is an isomorphism, but f is NOT an isomorphism.

Definition 4.11 (Inverse Image Sheaf)**.** *Let $f : X \to Y$ continuous, $\mathscr{G}$ a sheaf on Y. $U \subseteq X$ open, define $pre - f^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V)$.*

We have maps $\mathscr{G}(V) \to pre - f^{-1}\mathscr{G}(U)$, $s \mapsto f^{-1}s$, $f(U) \subseteq V$. $pre - f^{-1}\mathscr{G}(U) = \{f^{-1}s | s \in \mathscr{G}(V), V \supseteq f(U)\}$. $f^{-1}s = 0 \iff s|_W = 0$ where $f(U) \subseteq W \subseteq V$. This is a presheaf on X. We define $f^{-1}\mathscr{G}$ to be the sheafification of this presheaf.

Special Case: $X \subseteq Y$ is a subset, $i : X \to Y$ the inclusion, $\mathscr{G}|_X = i^{-1}\mathscr{G}$.

Example: $X \subseteq Y$ open, $pre - i^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq U} \mathscr{G}(V) = \mathscr{U}$. It is already a sheaf, and so the special case just mentioned above holds.

Exercise: $(f^{-1}\mathscr{G})_p = \mathscr{G}_{f(p)}$.

Adjoint Property

$f : X \to Y$ continuous, $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y, let $\varphi : \mathscr{G} \to f_*\mathscr{F}$ be a morphism of sheaves on Y. $U \subseteq X$ open, $V \supseteq f(U)$, then $\mathscr{G}(V) \overset{\varphi_V}{\to} \mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$ and maps are compatible with restrictions of $\mathscr{G}$, so this induces $\psi_U : pre{-}f^{-1}\mathscr{G}(U) \to \mathscr{F}(U)$, which gives a morphism $\psi : pre{-}f^{-1}\mathscr{G} \to \mathscr{F}$, we sheafify to get $\psi^+ : f^{-1}\mathscr{G} \to \mathscr{F}$.

Exercise: $\hom(\mathscr{G}, f_*\mathscr{F}) \to \hom(f^{-1}\mathscr{G}, \mathscr{F}) : \varphi \mapsto \psi^+$ is an isomorphism of abelian groups.

Category Theory Interpretation: f^{-1} is a left adjoint functor to f_* and f_* is a right adjoint functor to f^{-1}.

Let X be a variety (ringed-space)

Definition 4.12 ($\mathscr{O}_X$-module)**.** *An $\mathscr{O}_X$-module is a sheaf $\mathscr{F}$ on X such that $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module for all open $U \subseteq X$, such that if $V \subseteq U$ open then $\mathscr{F}(U) \to \mathscr{F}(V)$ is $\mathscr{O}_X(U)$-homomorphism (ie, $f \in \mathscr{O}_X(U)$, $m \in \mathscr{F}(U)$, $(f \cdot m)|_V = f|_V \cdot m|_V$.)*

An $\mathscr{O}_X$-homomorphism $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves such that $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is an $\mathscr{O}_X(U)$-homomorphism for all $U \subseteq X$ open.

Note $\ker\varphi \subseteq \mathscr{F}$ and $\operatorname{Im}\varphi \subseteq \mathscr{G}$ are sub $\mathscr{O}_X$-modules, $\mathscr{F}$ is an $\mathscr{O}_X$-module implies that $\mathscr{F}_P$ is an $\mathscr{O}_{X,P}$-module.

Definition 4.13 (Tensor Product)**.** *$\mathscr{F}, \mathscr{G}$, $\mathscr{O}_X$-modules, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} = [U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)]^+$*

Exercise: $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_p = \mathscr{F}_p \otimes_{\mathscr{O}_{X,P}} \mathscr{G}_p$

Definition 4.14 (Locally Free)**.** *An $\mathscr{O}_X$-module $\mathscr{F}$ is locally free if $\exists \bigcup_\alpha U_\alpha = X$ an open cover such that $\mathscr{F}|_{U_\alpha} \simeq \mathscr{O}_{U_\alpha}^{\otimes r}$.*

Example: $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, let $m \in \mathbb{Z}$.

Definition 4.15. *A sheaf $\mathscr{O}(m) = \mathscr{O}_{\mathbb{P}^n}(m)$ of $\mathscr{O}_{\mathbb{P}^n}$-modules is $\Gamma(U, \mathscr{O}(m)) = \{h \in k[\pi^{-1}(U)] : h(\lambda x) = \lambda^m h(x) \forall x \in \pi^{-1}(U), \lambda \in k^*\}$.*

Note: $f \in S = k[x_0, \ldots, x_n]$ homogeneous of degree > 0. $k[\pi^{-1}(D_+(f))] = k[D(f)] = S_f$. So $\Gamma(D_+(f), \mathscr{O}(m)) = (S_f)_m$.

$\mathscr{O}(m)$ is an invertible $\mathscr{O}_{\mathbb{P}^n}$-module (invertible means locally free of rank 1).

On $U_i = D_+(x_i) : \mathscr{O}_{U_i} \to \mathscr{O}(m)|_{U_i}$, $h \mapsto x_i^m h$. Note that $\mathscr{O}_{\mathbb{P}^n}(0) = \mathscr{O}_{\mathbb{P}^n}$. $\Gamma(\mathbb{P}^n, \mathscr{O}(m)) = S_m$ when $m \geq 0$ and 0 else.

Lemma 4.3. *$\mathscr{O}(m) \otimes_{\mathbb{P}^n} \mathscr{O}(p) \simeq \mathscr{O}(m+p)$.*

Proof. $U \subseteq \mathbb{P}^n$ open. $\Gamma(U, \mathscr{O}(m)) \otimes \Gamma(U, \mathscr{O}(p)) \to \Gamma(U, \mathscr{O}(m+p))$, $f \otimes g \mapsto fg$. Sheafification gives a map $\mathscr{O}(m) \otimes \mathscr{O}(p) \to \mathscr{O}(m+p)$, restricting to U we have $\mathscr{O}_{U_i} \otimes \mathscr{O}_{U_i} \to \mathscr{O}_{U_i}$. □

Consequence: $\mathscr{O}(m) \simeq \mathscr{O}(p)$ iff $m = p$. If $m \le p$ and $\mathscr{O}(m) \simeq \mathscr{O}(p)$ implies that $\mathscr{O}(m-p) \simeq \mathscr{O}(p-p)$, on the right we have nonzero global sections, on the left we only do if $m-p$ nonnegative, so $m = p$.

Later: Any invertible sheaf on $\mathbb{P}^n$ is isomorphic to $\mathscr{O}(m)$, $m \in \mathbb{Z}$.

Coherent Sheaves

Let X be an affine variety and $A = k[X]$, let M be an A-module.

Definition 4.16 (Quasi-Coherent Sheaf)**.** $\tilde{M} = [U \mapsto M \otimes_A \mathscr{O}_X(U)]^+$ *is called a quasi-coherent* $\mathscr{O}_X$*-module.*

Note: $M \otimes_A \mathscr{O}_X(D(f)) = M \otimes_A A_f = M_f$.

Examples $\tilde{A} = \mathscr{O}_X$, $Y \subseteq X$ closed, $I = I(Y) \subset A$. $\tilde{I} = \mathscr{I}_Y \subset \mathscr{O}_X$.

Exercise: $\tilde{A/I} = i_* \mathscr{O}_Y$.

Claim: $\tilde{M}_p = M_{I(p)}$ for $p \in X$, $I(p) = I(\{p\}) \subset A$.

$M_{I(p)} \to \tilde{M}_p$ by $m/f \mapsto (m \otimes 1/f)_p$, this tensor product is in $\tilde{M}(D(F))$. Surjectivity is clear.

Injective: if $m/f \mapsto 0 \in \tilde{M}_p$ then $(m \otimes 1/f)|_{D(h)} = 0$ for some $h \in A \setminus I(p)$, so $m \otimes 1/f = 0 \in M \otimes_A A_h = M_h$, thus $h^n m = 0 \in M$, and so $m/f = 0 \in M_{I(p)}$.

Consequence $\tilde{M}(U)$ =set of functions $s : U \to \coprod_{p \in U} M_{I(p)}$ such that $\forall p \in U$, there exists $p \in V \subset U$ and $m \in M, f \in A$ such tthat $s(p) = m/f \in M_{I(q)}$ for all $q \in V$.

Proposition 4.4. *$\tilde{M}(D(f)) \simeq M_f$.*

Corollary 4.5. *1. $\mathscr{O}_X(D(f)) = A_f$*

2. $\Gamma(X, \tilde{M}) = M$.

We now prove the proposition.

Proof. $\psi : M_f \to \tilde{M}(D(f))$ the obvious m

It is injective, as if $\psi(m/f^n) = 0$ then $m/f^n = 0 \in M_{I(p)}$ for all $p \in D(f)$. Thus, for all $p \in D(F)$, there exists $h_p \in A \setminus I(p)$ with $h_p m = 0 \in M$. $D(f) \subseteq \bigcup D(h_p) \Rightarrow V(f) \supseteq V(\{h_p\})$, so $f \in I(V(\{h_p\}))$ means $f^N = \sum_p a_p h_p$ for $a_p \in A$. $f^N m = \sum a_p h_p m = 0$, thus $m/f^n = 0 \in M_f$.

Surjectivity: Let $s \in \tilde{M}(D(f))$, there exists a cover $D(f) = \bigcup_{i=1}^r V_i$, $m_i \in M$, $h_i \in A$ such that $s = m_i/h_i$ on V_i. WLOG $V_i = D(g_i)$ and $V_i = D(h_i)$ (replace $m_i \mapsto g_i m_i$ by $h_i \mapsto g_i h_i$).

On $D(h_i h_j) = D(h_i) \cap D(h_j)$, $s = m_i/h_i = m_j/h_j \in \tilde{M}(D(h_i h_j))$, injectivity for $D(h_i h_j) : m_i/h_i = m_j/h_j \in M_{h_i h_j}$. So $(h_i h_j)^N (h_j m_i - h_i m_j) = 0 \in M$. Replace $m_i \mapsto h_i^N m_i$, $h_i \mapsto h_i^{N+1}$, $h_j m_i = h_i m_j$.

$D(f) \subseteq \bigcup D(h_i)$ so $f^n = \sum a_i h_i$ where $a_i \in A$. Set $m = \sum a_i m_i \in M$. Claim: $s = \psi(m/f^n)$.

For all j, $h_j m = \sum_i h_j a_i m_i = (\sum_i a_i h_i) m_j = f^n m_j$.

So $m/f^n = m_j/h_j = s$ on $D(h_j)$. □

Definition 4.17 (Quasi-Coherent and Coherent)**.** *Let X be any variety. An $\mathscr{O}_X$-mdule $\mathscr{F}$ is quasi-coherent if there exists an open affine cover $X = \bigcup U_i$ and $k[U_i]$-modules M_i such that $\mathscr{F}|_{U_i} \simeq \tilde{M}_i$ as $\mathscr{O}_{U_i}$-modules.*

$\mathscr{F}$ is coherent if M_i finitely generated $k[U_i]$-modules for all i.

Examples:

1. All locally free $\mathscr{O}_X$-modules are coherent, $U = \operatorname{Spec}-m(A)$, $\tilde{A^{\otimes r}} = \mathscr{O}_U^{\otimes r}$.
2. $Y \subseteq X$ closed, $\mathscr{I}_Y$ and $i_*\mathscr{O}_Y$ are coherent $\mathscr{O}_X$-modules.

Example: $X = \mathbb{A}^1$, $\mathscr{F}(U) = \begin{cases} \mathscr{O}_{\mathbb{A}^1}(U) & 0 \notin U \\ 0 & 0 \in U \end{cases}$. This is called the extension of the $\mathscr{O}_{\mathbb{A}^1\setminus\{0\}}$ by zeros. $\mathscr{F}$ is an $\mathscr{O}_X$-module, but is not quasi-coherent, as if $U \subseteq X$ is open affine and $0 \in U$, then $\Gamma(U, \mathscr{F}) = 0$ but $\mathscr{F}|_U$ is not the zero sheaf.

Exercises:

1. $\mathscr{F}$ is quasi-coherent iff $\mathscr{F}|_U \simeq \tilde{\mathscr{F}(U)}$ for all open affine $U \subseteq X$.
2. $\mathscr{F}$ is coherent implies $\mathscr{F}(U)$ is finitely generated $k[U]$-module for all affine open U.
3. $f : X \to Y$ a morphism of varieties.
   (a) f affine implies that $f_*\mathscr{O}_X$ quasi-coherent.
   (b) f finite $f_*\mathscr{O}_X$ coherent.

Example: M a finitely generated A-module, $X = \operatorname{Spec}-m(A)$. Then $\tilde{M}$ is locally free $\mathscr{O}_X$-module of rank r iff M is a projective A-module of const. rank r.

($\tilde{M}$ loc free iff $X = \bigcup D(f_i)$, $\tilde{M}_{D(f_i)} \simeq \mathscr{O}_{D(f_i)}^{\otimes} r$ iff $M_{f_i} = A_{f_i}^{\otimes r}$ iff M projective.)

Recall: X is complete implies $\Gamma(X, \mathscr{O}_X) = k$. More general fact: $\mathscr{F}$ a coherent sheaf on a complete variety X then $\dim_k \Gamma(X, \mathscr{F}) < \infty$. We will use this without proof. (Projective case in Hartshorne, General in EGA.)

Note that $\Gamma(\mathbb{A}^1, \mathscr{O}_{\mathbb{A}^1}) = k[t]$ which has infinite dimension.

Pushforward, Pullback

Let $f : X \to Y$ be a morphism of varieties. $\mathscr{F}$ an $\mathscr{O}_X$-module, then $f_*\mathscr{F}$ is a $f_*\mathscr{O}_X$-module, we have ring homomorphism $f^* : \mathscr{O}_Y \to f_*\mathscr{O}_X$, and so $f_*\mathscr{F}$ is an $\mathscr{O}_Y$-module.

Let $\mathscr{G}$ be a $\mathscr{O}_Y$-module $f^{-1}\mathscr{G}$ is an $f^{-1}\mathscr{O}_Y$-module. $(f^{-1}h \cdot f^{-1}s = f^{-1}(hs))$

$\mathscr{O}_Y \to f_*\mathscr{O}_X$ is the same as $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ by the adjoint property.

Definition 4.18 (Pullback)**.** *Define $f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$, we call it the pullback.*

Examples

1. $f^*\mathscr{O}_Y = \mathscr{O}_X$.

2. $(f^*\mathscr{G})_p = (f^{-1}(G))_p \otimes_{(f^{-1}\mathscr{O}_Y)_p} \mathscr{O}_{X,p} = \mathscr{G}_{f(p)} \otimes_{\mathscr{O}_{Y,f(p)}} \mathscr{O}_{X,p}$.

3. $U \subseteq Y$ open, $i : U \to Y$ inclusion, $i^*\mathscr{G} = \mathscr{G}|_U \otimes_{\mathscr{O}_Y|_U} \mathscr{O}_U = \mathscr{G}|_U$.

Adjoint Property

$f : X \to Y$ a morphism of SWFs. $\mathscr{G}$ an $\mathscr{O}_Y$-module. We have $f^{-1}\mathscr{O}_Y$-homomorphism $f^{-1}\mathscr{G} \to f^*\mathscr{G}$ by $s \mapsto s \otimes 1$. This gives an $\alpha : \mathscr{O}_Y$-homomorphism $\mathscr{F} \to f_*f^*\mathscr{G}$.

Lemma 4.6. *$\mathscr{G}$ is an $\mathscr{O}_Y$-module, $\mathscr{F}$ is an $\mathscr{O}_X$-module. Then* $\hom_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \simeq \hom_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F})$.

Proof. Given $\psi : f^*\mathscr{G} \to \mathscr{F}$ we obtain $\phi : \mathscr{G} \xrightarrow{\alpha} f_*f^*\mathscr{G} \xrightarrow{f_*\psi} f_*\mathscr{F}$.

Given $\phi : \mathscr{G} \to f_*\mathscr{F}$, we obtain $\tilde{\phi} : f^{-1}\mathscr{G} \to \mathscr{F}$ which is an $f^{-1}\mathscr{O}_Y$-hom. Take $\psi : f^*\mathscr{G} = f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \to \mathscr{F}$ by $s \otimes h \mapsto h \cdot \tilde{\phi}(s)$. □

Functoriality $X \xrightarrow{f} Y \xrightarrow{g} Z$ morphism of SWFs. If $\mathscr{F}$ is a sheaf on X, $g_*(f_*\mathscr{F}) = (gf)_*\mathscr{F}$.

Proposition 4.7.

1. *$\mathscr{G}$ on Z implies that $(gf)^{-1}\mathscr{G} = f^{-1}(g^{-1}\mathscr{G})$*

2. *$\mathscr{G}$ an $\mathscr{O}_Z$-module implies that $(gf)^*\mathscr{G} = f^*(g^*\mathscr{G})$.*

Proof. We will prove case 2.

$\mathrm{id} : (gf)^*\mathscr{G} \to (gf)^*\mathscr{G}$ gives $\mathscr{G} \to (gf)_*(gf)^*\mathscr{G} = g_*f_*(gf)^*\mathscr{G}$ gives $g^*\mathscr{G} \to f_*(gf)^*\mathscr{G}$ which gives $f^*(g^*\mathscr{G}) \to (gf)^*\mathscr{G}$.

We have a global homomorphism, so enough to check stalks. $f^*(g^*\mathscr{G})_p = (g^*\mathscr{G})_{f(p)} \otimes_{\mathscr{O}_{Y,f(p)}} \mathscr{O}_{X,p}$. This is $(\mathscr{G}_{g(f(p))} \otimes_{\mathscr{O}_{Z,gf(p)}} \mathscr{O}_{Y,f(p)}) \otimes_{\mathscr{O}_{Y,f(p)}} \mathscr{O}_{X,p} = \mathscr{G}_{gf(p)} \otimes_{\mathscr{O}_{Z,gf(p)}} \mathscr{O}_{X,p} = ((gf)^*\mathscr{G})_p$ □

Let $f : X \to Y$ be a morphism of SWFs. $\mathscr{G}$ is an $\mathscr{O}_Y$-module, and $f^{-1}\mathscr{G}$ is an $f^{-1}\mathscr{O}_Y$-module.

Definition 4.19 (Pullback)**.** *$f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$ is an $\mathscr{O}_X$-module.*

$f^* : \mathscr{O}_Y \to f_*\mathscr{O}_X$ induces a map $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$.

Some sections: $\sigma \in \mathscr{G}(V)$, we set $f^*\sigma = f^{-1}\sigma \otimes 1 \in \Gamma(f^{-1}(V), f^*\mathscr{G})$. The stalks $(f^*\mathscr{G})_p = \mathscr{G}_{f(p)} \otimes_{\mathscr{O}_{Y,f(p)}} \mathscr{O}_{X,p}$.

If $Z \xrightarrow{g} X \xrightarrow{f} Y$, then $(fg)^*\mathscr{G} = g^*(f^*\mathscr{G})$.

Corollary 4.8. *If $\mathscr{G}$ is a locally free $\mathscr{O}_Y$-module, then $f^*\mathscr{G}$ is a locally free $\mathscr{O}_X$-module of the same rank.*

Proof. Let $Y = \bigcup V_i$ be an open cover such that $\mathscr{G}|_{V_i} \simeq \mathscr{O}_{V_i}^{\oplus r}$. Set $U_i = f^{-1}(V_i) \subset X$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ p \uparrow & & \uparrow q \\ U_i & \xrightarrow{f'} & V_i \end{array}$$

$f^*\mathscr{G}|_{U_i} = p^*f^*\mathscr{G} = f'^*q^*\mathscr{G} \simeq f'^*q^*(\mathscr{O}_{V_i}^{\oplus r}) = \mathscr{O}_{U_i}^{\oplus r}$ □

Lemma 4.9. *$f : X \to Y$ a morphism, $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ is a short exact sequence of $\mathscr{O}_Y$-modules. Then $f^*\mathscr{G}' \to f^*\mathscr{G} \to f^*\mathscr{G}'' \to 0$ is an exact sequence of $\mathscr{O}_X$-modules. If $\mathscr{G}''$ is locally free, then the first map is injective.*

Proof. On stalks we start with $0 \to \mathscr{G}'_{f(p)} \to \mathscr{G}_{f(p)} \to \mathscr{G}''_{f(p)} \to 0$ exact. Tensor produce is right exact and gets us to f^*, ad so we have the first part of the theorem immediately.

If $\mathscr{G}''$ is locally free, then $\mathscr{G}''_{f(p)}$ is a free $\mathscr{O}_{Y,f(p)}$-module, and so the original sequence is split-exact. □

Definition 4.20 (Generated by Finitely Many Global Sections)**.** *The $\mathscr{O}_X$-module $\mathscr{F}$ is generated by finitely many global sections iff $\exists$ a surjective map $\mathscr{O}_X^{\oplus m} \to \mathscr{F}$.*

Equivalently, $\exists s_1, \dots, s_m \in \Gamma(X, \mathscr{F})$ such that $\mathscr{F}_p$ generated by $(s_1)_p, \dots, (s_m)_p$ as an $\mathscr{O}_{X,p}$-module.

Example: Any quasi-coherent $\mathscr{O}_X$-module, if X is affine (this is just generated by global sections, requires coherent to be generated by finitely many)

Example: $\mathscr{O}_{\mathbb{P}^n}(1)$ is generated by $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathscr{O}(1))$.

Suppose that $f : X \to \mathbb{P}^n$ is a morphism, then $\mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathscr{O}(1) \to 0$ exact implies that $\mathscr{O}_X^{\oplus n+1} \to f^*\mathscr{O}(1) \to 0$ is exact, so $f^*\mathscr{O}(1)$ is generated by global sections $f^*(x_0), \dots, f^*(x_n)$.

Proposition 4.10. *X a variety, $\mathscr{L}$ invertible $\mathscr{O}_X$-module generated by global sections $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$. Then $\exists! f : X \to \mathbb{P}^n$ such that $f^*\mathscr{O}(1) \simeq \mathscr{L}$ and $f^*(x_i) \leftrightarrow s_i$.*

Proof. Set $U_i = \{p \in X | (s_i)_p \notin \mathfrak{m}_p\mathscr{L}_p\}$.

U_i open: If $V \subseteq X$ open with $\mathscr{L}|_V \simeq \mathscr{O}_V$, then $\mathscr{L}|_V$ is generated by $t \in \Gamma(V, \mathscr{L})$ so we write $s_i = h_i t$ on V with $h_i \in k[V]$. Then $U_i \cap V = \{p \in V | h_i(p) \neq 0\}$ is open in V.

Note: $\mathscr{L}$ is generated by $s_0, \dots, s_n$ implies that $X = \bigcup_{i=1}^n U_i$, and $\mathscr{O}_{U_i} \simeq \mathscr{L}|_{U_i}$ implies that $1 \mapsto s_i|_{U_i}$.

On U_i, we can write $s_i = h_{ij}s_j$ for some $h_{ij} \in k[U_i]$. We define a map $g : U_i \to \mathbb{P}^n$ by $f(p) = (h_{i0}(p) : \dots : h_{in}(p))$.

The maps are compatible: On $U_i \cap U_j$, $h_{\ell j}s_\ell = s_j = h_{ij}s_i = h_{ij}h_{\ell i}s_\ell$, so $h_{\ell j} = h_{\ell i}h_{ij}$. The map on $U_\ell : p \mapsto (h_{\ell 0}(p) : \dots : h_{\ell n}(p)) = h_{\ell i}(p)h_{i0}(p) : \dots : h_{\ell i}(p)h_{in}(p)$. Thus, we have a morphism $f : X \to \mathbb{P}^n$.

Claim: $\exists$ isomorphism $\mathscr{L} \to f^*\mathscr{O}(1)$ by $s_i \mapsto f^*(x_i)$. On U_i, we define $\mathscr{L}|_{U_i} \to f^*\mathscr{O}(1)|_{U_i}$ by $hs_i \mapsto hf^*(x_i)$. This means that $s_\ell = h_{i\ell}s_i \mapsto h_{i\ell}f^*(x_i)$, we must check that $f^*(x_\ell) = h_{i\ell} f^*(x_i)$. The definition of f implies that $(x_\ell/x_i) \circ f = \frac{h_{i\ell}}{h_{ii}} = h_{i\ell}$, so $f^*(x_\ell) = f^*(\frac{x_\ell}{x_i} x_i) = f^*(\frac{x_\ell}{x_i}) f^*(x_i) = h_{i\ell} f^*(x_i)$.

And now we prove uniqueness: If $f : X \to \mathbb{P}^n$ is any morphism such that $\mathscr{L} \simeq f^*\mathscr{O}(1)$ and $s_i \leftrightarrow f^*(x_i)$ on U_i, $h_{i\ell} s_i = f^*(x_\ell) = f^*(\frac{x_\ell}{x_i} x_i) = f^*(\frac{x_\ell}{x_i}) f^*(x_i) = f^*(\frac{x_\ell}{x_i}) s_i$, so $f^*(x_\ell/x_i) = h_{i\ell}$ on U_i. □

Definition 4.21 (Very Ample Sheaf)**.** *Let $\mathscr{L}$ be an invertible sheaf on X. $\mathscr{L}$ is very ample iff $\mathscr{L}$ is generated by (finitely many) global sections and the map $f : X \to \mathbb{P}^n$ given by generators $s_0, \ldots, s_n \in \Gamma(X, \mathscr{L})$ is an isomorphism $f : X \to W \subseteq \mathbb{P}^n$ locally closed.*

Exercise: $\mathscr{O}_{\mathbb{P}^n}(m)$ is very ample iff $m \geq 1$.

Definition 4.22 (PGL)**.** *$PGL(n) = GL(n+1)/k^*$.*

Exercise: $PGL(n)$ is an affine algebraic group.
FACT: Every invertible sheaf on $\mathbb{P}^n$ is isomorphic to $\mathscr{O}(m)$ for some m.

Corollary 4.11. $\mathrm{Aut}(\mathbb{P}^n) \simeq PGL(n)$.

Proof. $PGL(n) \leq \mathrm{Aut}(\mathbb{P}^n)$ is trivial.

Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be an automorphism. The fact implies that $f^*\mathscr{O}(1) \simeq \mathscr{O}(m)$ for some m. In fact, $\binom{n+m}{m} = \dim_k \Gamma(\mathbb{P}^n, \mathscr{O}(m)) = \dim \Gamma(f^*\mathscr{O}(1)) = \dim \Gamma(\mathscr{O}(1)) = n+1$, so $m = 1$.

$f^*x_0, \ldots, f^*x_n \in \Gamma(\mathbb{P}^n, \mathscr{O}(1))$ form a basis. We write $f^*(x_i) = \sum_{i=0}^n a_{ij} x_j$ for $a_{ij} \in k$. Then $A = (a_{ij}) \in GL(n+1)$.

Define $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ by $\varphi(x_0 : \ldots : x_n) = (\sum a_{0j} x_j : \ldots : a_{nj} x_j)$. $\varphi^*(\frac{x_\ell}{x_i}) = \frac{\sum a_{\ell j} x_j}{\sum a_{ij} x_j} = f^*(\frac{x_\ell}{x_i})$.

Thus, $f = \varphi \in PGL(n)$. □

Corollary 4.12. *$\mathbb{P}^n$ is not an algbriac group.*

Normal Varieties

Definition 4.23 (Normal Variety)**.** *X is irreducible. Then X is normal iff $\mathscr{O}_{X,P}$ is normal (integrally closed) for all p.*

Example: Nonsingular varieties.
Note: X affine, then X is normal iff $k[X]_{\mathfrak{m}}$ normal for all maximal $\mathfrak{m}$ iff $k[X]$ is normal.
Exercise: If A is a domain, $S \subseteq A$ multiplicative, then $\overline{S^{-1}A} = S^{-1}\overline{A}$.

Definition 4.24 (Normalization)**.** *If f is an affine variety, $k[X] \subset k(X)$, then $\overline{k[X]} \subseteq k(X)$ is the integral closure. The normalization of X is $\overline{X} =$ $\mathrm{Spec}-m(\overline{k[X]})$.*

As we have the inclusion $k[X] \to \overline{k[X]}$, we get a projection map $\overline{X} \to X$ which is finite.

Lemma 4.13. *$\varphi : U \to X$ morphism of affines, φ an open embedding iff $\exists f_1, \dots f-n \in k[X]$ such that $(\varphi^* f_1, \dots, \varphi^* f_n) = (1) \subset k[U]$ and $\varphi^* : k[X]_{f_i} \to k[U]_{\varphi^* f_i}$ is an isomorphism for all i.*

Proof. $\Rightarrow$: Take open cover $U = \bigcup_{i=1}^r D(f_i)$, $f_i \in k[X]$.

$\Leftarrow$: Set $V = \bigcup_{i=1}^r D(f_i) \subseteq X$. $(\varphi^* f_1, \dots, \varphi^* f_r) = (1) \subseteq k[U]$ implies that $\varphi(U) \subseteq V$. Thus, we have that $\varphi : \varphi^{-1}(D(f_i)) \to D(f_i)$ is an isomorphism, so $\varphi : U \to V$ isomorphism. □

Lemma 4.14. *Assume X affine, $U \subseteq X$ is an open affine, then $\bar{U} \subseteq \bar{X}$ open affine.*

Proof. $k[X] \subseteq k[U] \subseteq k(X)$. Thus, $\overline{k[X]} \subseteq \overline{k[U]}$, so we have a morphism $\varphi : \bar{U} \to \bar{X}$. Take $f_1, \dots, f_n \in k[X]$ as in lemma wrt $U \subset X$.

$(f_1, \dots, f_n) = (1) \subseteq \overline{k[U]}$, and $k[\bar{U}]_{f_i} = \overline{k[U]_{f_i}} = \overline{k[X]_{f_i}} = k[\bar{X}]_{f_i}$. And so, the lemma implies that φ is an open embedding. □

Exercise: Given pre-varieties $X_1, \dots, X_n$, open subsets $U_{ij} \subseteq X_i$ and isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ such that $U_{ii} = X_i$, $\varphi_{ii} = \text{id}$, for all i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi ij$ on $U_{ij} \cap U_{ik}$, then $\exists!$ prevariety X with morphisms $\psi_i : X_i \to X$ such that $\psi_i : X_i \to \text{open} \subset X$ is an isomorphism, $X = \bigcup_{i=1}^n \psi_i(X_i)$, $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

Definition 4.25 (Normal)**.** *A variety X is normal iff X is irreducible and $\mathscr{O}_{X,P}$ are normal for all p.*

Construction: X an irreducible variety, $X = X_1 \cup \dots \cup X_n$ open affine cover, set $U_{ij} = X_i \cap X_j$. Then U_{ij} is affine, have $\psi_{ij} : U_{ij} \to U_{ji}$ nbe the identity. Now $\bar{U}_{ij} \subseteq \bar{X}_i$ and still have $\phi_{ij} : \bar{U}_{ij} \to \bar{U}_{ji}$ is the identity, so the satisfy the hypotheses of the exercise. Thus, there exists a prevariety $\bar{X} = \bar{X}_1 \cup \dots \cup \bar{X}_n$. We call this the normalization of X.

Note: $\overline{k[X_i]}$ is a finitely generated $k[X_i]$-module, so we have finite $\pi : \bar{X}_i \to X_i$, which we can glue to a morphism $\pi : \bar{X} \to X$.

Exercise: $\pi : \bar{X} \to X$ is finite. (Check that $\pi^{-1}(X_i) = \bar{X}_i$)

Exercise: $\varphi : X \to Y$ affine morphism of pre-varieities. Then Y is separated implies that X is serpated, and so $\bar{X}$ is an irreducible normal separated variety.

Example: X irred curve implies $\pi : \bar{X} \to X$ resoluton of singularities.

Definition 4.26 (Local Ring along Subvariety)**.** *Let X be a variety, $V \subseteq X$ irreducible and closed. Then $\mathscr{O}_{X,V} = \varinjlim_{U \subseteq X, U \cap V \neq} \mathscr{O}_X(U)$.*

If X is irreducible, then $\mathscr{O}_{X,V} = \{f \in k(X) : f \text{ is defined at one point of } V\}$.

General Case: $U \subseteq X$ open affine, $U \cap V \neq \emptyset$, $P = I(U \cap V) \subseteq k[U]$, $\mathscr{O}_{X,V} = k[U]_P$.

Definition 4.27 (Regular along V)**.** *X is regular along V if $\mathscr{O}_{X,V}$ is a regular local ring. i.e., the maximal ideal in $\mathscr{O}_{X_V}$ is generated by $\dim(\mathscr{O}_{X,V}) = \operatorname{codim}(V; X)$ elements. This happens iff $V \not\subseteq X_{sing}$.*

5 Divisors

Let X be a normal variety.

Definition 5.1 (Prime Divisor). *A prime divisor on X is a closed irreducible subvariety of codimension 1.*

Note: Y a prime divisor implies that $\mathscr{O}_{X,Y}$ is a normal Nötherian local ring of dimension 1. That is, $\mathscr{O}_{X,Y}$ is a DVR.

Consequences

1. $\operatorname{codim}(X_{sing}; X) \geq 2$.
2. Have valuation map $v_Y : k(X)^* \to \mathbb{Z}$ for each prime divisor $Y \subseteq X$.

Lemma 5.1. *Let $f \in k(X)^*$. Then $v_Y(f) = 0$ for all but finitely many Y.*

Proof. Show $v_Y(f) < 0$ for finitely many Y. Set $U \subseteq X$ open set where f defined. $Z = X \setminus U$. $v_Y(f) < 0$ iff $f \notin \mathscr{O}_{X,Y}$ iff f is not defined at any point of Y iff $Y \subseteq Z$ component. □

Definition 5.2 (Divisor Group). *Define* $\operatorname{Div}(X)$ =*free abelian group generated by all prime divisors.*

An element $D = \sum n_i[Y_i]$ is a finite sum and is called a Weil Divisor.

Definition 5.3. *For $f \in k(X)^*$, set $(f) = \sum_Y v_Y(f) \cdot [Y] \in \operatorname{Div}(X)$.*

Note $(f^{-1}) = -(f)$, $(fg) = (f) + (g)$ Thus $k(X)^* \to \operatorname{Div}(X)$ by $f \mapsto (f)$ is a group homomorphism.

Definition 5.4 (Class Group). *Define* $\mathrm{C}\ell(X) = \operatorname{Div}(X)/\{(f) : f \in k(X)^*\}$.

Example: $\mathrm{C}\ell(\mathbb{A}^n) = 0$, as every hypersurface corresponds to a prime divisor.

Remark: X complex, $\dim(X) = n$, then $\mathrm{C}\ell(X) \leftrightarrow H_{2n-2}(X; \mathbb{Z})$.

Remark: X irreducible but not normal, we can still define $\mathrm{C}\ell(X)$, use $v_Y(f/g) = length_{\mathscr{O}_{X,Y}}(\mathscr{O}_{X,Y}/(f)) - length_{\mathscr{O}_{X,Y}}(\mathscr{O}_{X,Y}/(g))$.

Divisors on $\mathbb{P}^n$

Note: All prime divisors are hypersurfaces $Y = V_+(h)$ where $h \in S = k[x_0, \ldots, x_n]$ is an irreducible form.

Definition 5.5. *Degree of a Divisor* $\deg : \operatorname{Div}(\mathbb{P}^n) \to \mathbb{Z}$ *by* $\deg(\sum m_i[Y_i]) = \sum m_i \deg Y_i$.

Let $f \in k(\mathbb{P}^n)^*$, $m_i \in \mathbb{Z}$, $g = \prod_{i=1}^r h_i^{m_i}$, $h_i \in S$ irreducible form. Then $\sum m_i \deg(h_i) = 0$, so $Y_i = V_+(h_i) \subseteq \mathbb{P}^n$ is a prime divisor, so $v_{Y_i}(h_i) = 1$, $v_{Y_i}(f) = m_i$.

$(f) = \sum_{i=1}^r m_i[Y_i]$ implies that $\deg(f) = \sum m_i \deg(h_i) = 0$, thus, $\deg : \mathrm{C}\ell(\mathbb{P}^n) \to \mathbb{Z}$ is well-defined.

Claim: Isomorphism.

Surjective: $H \subseteq \mathbb{P}^n$ hyperplane, $\deg(m[H]) = m$. Injective: Let $D = \sum m_i[Y_i] \in \mathrm{Div}(\mathbb{P}^n)$, suppose $\deg(D) = 0$, then $Y_i = V_+(h_i)$, $h_i \in S$ irred form, so $\sum m_i \deg(h_i) = \deg(D) = 0$, so $f = \prod h_i^{m_i} \in k(\mathbb{P}^n)^*$ and $D = (f)$.

Later: X nonsingular implies $\mathrm{C}\ell(X) \simeq \mathrm{Pic}(X)$, thus $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$.

X normal, $Y \subseteq X$ prime divisor implies that $\mathscr{O}_{X,Y}$ is a DVR.

Theorem 5.2. *R normal Nötherian domain implies $R = \cap_{\mathrm{ht}\,\mathfrak{p}=1} R_\mathfrak{p}$ where $\mathrm{ht}\,\mathfrak{p} = \dim R_\mathfrak{p}$, the max m such that $\exists 0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_m = \mathfrak{p}$.*

Corollary 5.3. *If X is normal, $f \in k(X)^*$, then $f \in k[X]$ iff $v_Y(f) \geq 0$ for all $Y \subseteq X$ prime divisors.*

Lemma 5.4. *R Nötherian, then R is a UFD iff all prime ideals of height one are principal.*

Proof. $\Rightarrow$: Assume R a UFD. $P \subseteq R$ prime of height 1, let $x \in P$ be an irreducible element, $0 \subsetneq (x) \subseteq P$, so $P = (x)$.

$\Leftarrow$: Exercise: R any Nötherian domain then every element of R is a product of irreducible elements.

Unique Factorization: Show $x \in R$ irred and $x|fg$ implies that $x|f$ or $x|g$, ie, $(x) \subseteq R$ is prime, let $P \supseteq (x)$ min prime, PIT implies $\mathrm{ht}(P) = 1$ implies $P = (y)$, $x = ay$, $a \in R$ a unit. □

Proposition 5.5. *X irreducible affine variety, $k[X]$ a UFD iff X normal and $\mathrm{C}\ell(X) = 0$.*

Proof. $\Rightarrow$: UFD implies normal. Let $Y \subseteq X$ a prime divisor, $P = I(Y) \subseteq k[X]$ prime of height 1. $P = (h) \subseteq k[X]$, $h \in k[X]$. So $(h) = [Y]$ implies $[Y] = 0 \in \mathrm{C}\ell(X)$.

$\Leftarrow$: Let $P \subseteq k[X]$ prime of height 1, $Y = V(P) \subseteq X$ a prime divisor, $[Y] = 0 \in \mathrm{C}\ell(X)$ so $[Y] = (h) \in \mathrm{Div}(X)$, $h \in k(X)^*$. $v_Z(h) \geq 0$ for all $Z \subseteq X$ prime divisors implies $h \in k[X]$. Claim: $P = (h) \subseteq k[X]$. $\supseteq$ is clear. Let $g \in P$, then $v_Y(g) \geq 1$, so $v_Z(g/f) \geq 0$ for all Z, so $g/f \in k[X]$, and $g = af \in (f)$. □

Proposition 5.6. *X normal, $Z \subseteq X$ is a proper closed subset, $U = X \setminus Z$. Then*

1. *$\mathrm{C}\ell(X) \to \mathrm{C}\ell(U)$ by $[Y] \mapsto [Y \cap U]$ if $Y \cap U \neq \emptyset$ and 0 else is surjective*
2. *If $\mathrm{codim}(Z; X) \geq 2$, then $\mathrm{C}\ell(X) = \mathrm{C}\ell(U)$.*
3. *If Z prime divisor, then $\mathbb{Z} \to \mathrm{C}\ell(X) \to \mathrm{C}\ell(U) \to 0$ is exact.*

Proof.
1. Well defined $\mathrm{Div}(X) \to \mathrm{Div}(U)$. $f \in k(X)^*$, $(f) \mapsto (f|_U)$ (because if $Y \subseteq X$ is a prime divisor, $Y \cap U \neq \emptyset$ then $\mathscr{O}_{X,Y} = \mathscr{O}_{U,U\cap Y}$).

   Thus, $\mathrm{C}\ell(X) \to \mathrm{C}\ell(U)$ is well defined. Surjective: If $V \subseteq U$ a prime divisor, $\bar{V} \subseteq X$ is a prime divisor $[\bar{V}] \mapsto [V]$.
2. $\mathrm{Div}(X) = \mathrm{Div}(U)$, so $(f) = (f|_U)$.

3. If $D = \sum n_Y[Y] \mapsto 0 \in \mathrm{C}\ell(U)$, then $\exists f \in k(X)^*$: $v_Y(f) = n_Y$ for all $Y \neq Z$. $D - (f) = m[Z] \Rightarrow D = m[Z] \in \mathrm{C}\ell(X)$.

$\square$

Example: $X = V(xy - z^2) \subset \mathbb{A}^3$.

Exercise: X (above) is normal. $L = V(y) \cap X = V(y, z)$ is a prime divisor on X.

Max ideal of $\mathscr{O}_{X,L}$ is generated by z, $y = \frac{z^2}{x} \in \mathscr{O}_{X,L}$.

Set $U = X \setminus L$ affine, $k[U] = (k[x,y,z]/(xy - z^2))_y = k[y,z]_y$ is a UFD, so $\mathrm{C}\ell(U) = 0$. $\mathbb{Z} \to \mathrm{C}\ell(X) \to \mathrm{C}\ell(U) = 0$. So $\mathrm{C}\ell(X) = \{m[L] : m \in \mathbb{Z}\}$, $y \in k(X)^*$. $(y) = v_L(y)[L] = v_L(z^2)[L] = 2[L]$. Thus, $\mathrm{C}\ell(X) = \mathbb{Z}/2\mathbb{Z}$ or $\mathrm{C}\ell(X) = 0$. As $k[x,y,z]/(xy - z^2)$ is not a UFD, $\mathrm{C}\ell(X) = \mathbb{Z}/2\mathbb{Z}$.

Picard Group

Invertible $\mathscr{O}_X$-module = line bundle.

Let X be any variety, $\mathscr{L}_1$ and $\mathscr{L}_2$ are line bundles, then $\mathscr{L}_1 \otimes_{\mathscr{O}_X} \mathscr{L}_2$ is a line bundle. $\mathscr{L}$ is invertible implies that we can define $\mathscr{L}^{-1} = [U \mapsto \hom_{\mathscr{O}_U}(\mathscr{L}|_U, \mathscr{O}_U)]$.

Exercise: $\mathscr{L}^{-1}$ is an invertible $\mathscr{O}_X$-module and $\mathscr{L} \otimes \mathscr{L}^{-1} \simeq \mathscr{O}_X$.

Definition 5.6 (Picard Group)**.** $\mathrm{Pic}(X) = \{$*isomorphism classes of invertible sheaves on* $X\}$.

This is a group under tensor product.

Notation: X irreducible variety, $\mathscr{L}$ an invertible sheaf on X and $s \in \mathscr{L}(U)$, $t \in \mathscr{L}(V)$ are nonzero sections. Take $W \subset U \cap V$ open such that $\mathscr{L}|_W \simeq \mathscr{O}_W$ generated by $u \in \mathscr{L}(W)$. Then $s|_W = fu$ and $t|_W = gu$, $f, g \in k[W]$. Define $s/t = f/g \in k(X)^*$.

Example: $s_0, \ldots, s_n \in \Gamma(X, \mathscr{L})$. Define $f : X \dashrightarrow \mathbb{A}^n \subset \mathbb{P}^n$. $f(x) = (s_1/s_0(x), \ldots, s_n/s_0(x)) = (s_1/s_0(x) : \ldots : s_n/s_0(x) : 1)$. If $s_0, \ldots, s_n$ generate $\mathscr{L}$, then f extends to a morphism $f : X \to \mathbb{P}^n$.

X is a normal variety, $s \in \mathscr{L}(U)$, $s \neq 0$, $Y \subseteq X$ is a prime divisor, take $V \subseteq X$ open such that $\mathscr{L}|_V \simeq \mathscr{O}_V$ generated by $t \in \mathscr{L}(V)$ and $V \cap Y \neq \emptyset$.

Definition 5.7. $V_Y(s) = V_Y(s/t)$.

Well defined, if $V' \subseteq X$, $V' \cap Y \neq \emptyset$, $\mathscr{L}|_{V'}$ generated by $y' \in \mathscr{L}(V')$ implies that t/t' nowhere vanishing function on $V \cap V'$, so t/t' is a unit in $\mathscr{O}_{X,Y}$.

Thus, $V_Y(s/t') = V_Y(s/t \cdot t/t') = V_Y(s/t) + 0$.

Definition 5.8. $(s) = \sum_Y V_Y(s)[Y] \in \mathrm{Div}(X)$.

Note: If $s' \in \mathscr{L}(U')$ then $(s') = (s) + (s'/s)$. Then $(s') = (s) \in \mathrm{C}\ell(X)$, so all nonzero sections of a line bundle are equivalent in the class group.

Thus, we have a well defined map $\mathrm{Pic}(X) \to \mathrm{C}\ell(X)$ by $\mathscr{L} \mapsto (s)$.

Check: $s_1 \in \mathscr{L}_1(U)$, $s_2 \in \mathscr{L}_2(U)$, then $s_1 \otimes s_2 \in \mathscr{L}_1 \otimes \mathscr{L}_2(U)$ and $(s_1 \otimes s_2) = (s_1) + (s_2)$, so this map is a group homomorphism.

Cartier Divisors

X normal.

Definition 5.9 (Cartier Divisor)**.** *A Cartier is a Weil Divisor* $D = \sum n_i[Y_i]$ *which is locally principal. I.E. there exists an open covering* $\bigcup_{i=1}^n U_i = X$ *such that* $D|_{U_i} = 0 \in \mathrm{C}\ell(U_i)$ *for all* i*.*

Note: $D|_{U_j} = (f_j)$ where $f_j \in k(U_j)^*$. We can think about D as the collection $\{f_j\}$ of these generators.

Definition 5.10 (Cartier Class Group)**.** $\mathrm{CaC}\ell(X) = \{$*Cartier Divisors on* $X\}/\{(f) : f \in k(X)^*\}$*.*

Recall: $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module, $s \in \mathscr{L}(U)$ nonzero section, then $(s) = \sum_{Y \text{ prime}} v_Y(s) \cdot [Y] \in \mathrm{Div}(X)$ where $v_Y(s) = v_Y(s/t)$ for $t \in \mathscr{L}(V)$ generator of $\mathscr{L}|_V \simeq \mathscr{O}_V$, $Y \cap V \neq \emptyset$.

Note: (s) is Cartier.

Thus, we have a group homomorphism $\mathrm{Pic}(X) \to \mathrm{CaC}\ell(X) \subset \mathrm{C}\ell(X)$.

Line Bundles from Divisors

Let $D = \sum n_Y[Y] \in \mathrm{Div}(X)$.

Definition 5.11. $\mathscr{O}_X$*-module* $\mathscr{L}(D)$ *or* $\mathscr{O}_X(D)$ $\Gamma(U, \mathscr{L}(D)) = \{f \in k(X)^* | v_Y(f) \geq -n_Y$ *for all prime divisors* Y *such that* $Y \cap U \neq \emptyset\} \cup \{0\}$*.*

Example: $\mathscr{L}(0) = \mathscr{O}_X(0) = \mathscr{O}_X$.

Example: $X = \mathbb{P}^1$, $Q = (a : b) \in X$, $D = n[Q]$, $Q = V_+(h)$, $h = bx_0 - ax_1 \in k[x_0, x_1]$. So $\mathscr{O}_{\mathbb{P}^1}(n[Q]) \simeq \mathscr{O}_{\mathbb{P}^1}(n)$ by $f \mapsto h^n f$.

Note: If $h \in k(X)^*$ then $\mathscr{O}_X(D + (h)) \to \mathscr{O}_X(D)$ by $f \mapsto hf$ is an isomorphism. $v_Y(f) \geq -n_Y - v_Y(h) \iff v_Y(fh) \geq -n_Y$.

Consequence: D is a Cartier Divisor implies that $\mathscr{O}_X(D)$ is an invertible $\mathscr{O}_X$-module.

If $D|_U = (h) \in \mathrm{Div}(U)$, then $\mathscr{O}_X(D)|_U = \mathscr{O}_U((h)) \simeq \mathscr{O}_U$. Note, this says we have a map $\mathrm{CaC}\ell(X) \to \mathrm{Pic}(X)$ by $D \mapsto \mathscr{O}_X(D)$.

WARNING: If $f \in \Gamma(U, \mathscr{O}_X(D))$ then f a rational function. The notation (f) means two things!

Proposition 5.7. $\mathrm{Pic}(X) \simeq \mathrm{CaC}\ell(X)$ *as abstract groups.*

Proof. Will check that $\mathrm{Pic}(X) \leftrightarrows \mathrm{CaC}\ell(X)$ are inverse maps.

Let $D = \sum n_Y[Y]$ a Cartier Divisor. Set $V = X \setminus (\cup_{n_Y < 0} Y) \subseteq X$ open. Then $1 \in k(X)^*$ is a section of $\mathscr{O}_X(D)$ over V. ($v_Y(1) \geq -n_Y \iff n_Y \geq 0$).

Claim: $(1) = D \in \mathrm{Div}(X)$. If $D|_U = (g) \in \mathrm{Div}(U)$, then $\mathscr{O}_X(D)|_U \simeq \mathscr{O}_U$ gneerated by $h^{-1} \in \Gamma(U, \mathscr{O}_X(D))$, $Y \cap U \neq \emptyset$ implies that $v_Y(1) = v_Y(1/h^{-1}) = v_Y(h)$. Thus, $(1)|_U = (h)|_U = D|_U$.

Thus, $\mathrm{CaC}\ell(X) \to \mathrm{Pic}(X) \to \mathrm{CaC}\ell(X)$ is the identity.

Let $\mathscr{L}$ be a line bundle on X. $t \in \Gamma(U, \mathscr{L})$ a non-zero section.

Note: If $0 \neq s \in \mathscr{L}(V)$, then $Y \cap V \neq \emptyset$ implies $v_Y(s/t) = v_Y(s) - v_Y(t) \geq -v_Y(t)$, and so $s/t \in \Gamma(V, \mathscr{O}_X((t)))$.

Claim: $\mathscr{L} \simeq \mathscr{O}_X((t))$ by $s \mapsto s/t$. If $\mathscr{L}|_V \simeq \mathscr{O}_V$ gneerated by $u \in \mathscr{L}(V)$ then $(t)|_V = (t/u) \in \mathrm{Div}(V)$ implies that $\mathscr{O}_X((t))|_V$ is generated by u/t as $u \mapsto u/t$ we get $\mathscr{L}|_V \simeq \mathscr{O}_X((t))|_V$. ∎

Examples:

1. $\mathrm{Pic}(\mathbb{A}^n) = \mathrm{CaC}\ell(\mathbb{A}^n) \subset \mathrm{C}\ell(\mathbb{A}^n) = 0$, so all line bundles on $\mathbb{A}^n$ are trivial.

   Fact: Any locally free $\mathscr{O}_{\mathbb{A}^n}$-module of finite rank is trivial.

2. $\mathbb{P}^n = \bigcup_{i=0}^n D_+(x_i)$, $\mathrm{C}\ell(D_+(x_i)) = 0$, so all Weil divisors are Cartier, thus $\mathrm{Pic}(\mathbb{P}^n) = \mathrm{CaC}\ell(\mathbb{P}^n) = \mathrm{C}\ell(\mathbb{P}^n) = \mathbb{Z}$.

   By the maps we have, any line bundle is isomorphic to $\mathscr{O}_{\mathbb{P}^n}(m[H])$, where $H \subset \mathbb{P}^n$ is a hyperplane. $I(H) = (h)$, $\mathscr{O}_{\mathbb{P}^n}(m[H]) \simeq \mathscr{O}_{\mathbb{P}^n}(m)$ by $f \mapsto h^m f$. Thus, $\mathrm{Pic}(\mathbb{P}^n) = \{\mathscr{O}(m)\}$.

3. $X = V(xy - z^2) \subset \mathbb{A}^3$. $L = V(y) \cap X$, $I(L) = (y, z) \subset k[\mathbb{A}^3]$. Claim: $[L]$ is not Cartier. Otherwise there exists open affine $U \subset X$ such that $P = (0,0,0) \in U$ with $[L \cap U] = (f|_U) \in \mathrm{Div}(U)$. Thus $f \in k[U]$ and $I(L \cap U) = (f) \subset k[U]$, so $I(L) \cdot \mathscr{O}_{X,P} = (y, z) \subset \mathscr{O}_{X,P}$ is principal.

   But $P \in X$ is a singular point, so $\dim_k(\mathfrak{m}_P/(m)_P^2) = 3$, so $\{x, y, z\}$ is a basis, and so $\dim((y, z) + \mathfrak{m}_P^2/\mathfrak{m}_P^2) = 2$, so $(y, z) \subset \mathscr{O}_{X,P}$ is not principal, which is a contradiction. Thus $\mathrm{CaC}\ell(X) = \mathrm{Pic}(X) = 0 \neq \mathrm{C}\ell(X)$.

Note: $\mathscr{O}_{X,P}$ is not a UFD, as $(y, z) \subset \mathscr{O}_{X,P}$ is height 1 prime but not principal.

Definition 5.12 (Locally Factorial)**.** *An irreducible variety X is locally factorial if $\mathscr{O}_{X,P}$ is a UFD for all $p \in X$.*

Example: Nonsingular implies locally factorial implies normal.

Proposition 5.8. *X locally factorial implies* $\mathrm{Pic}(X) = \mathrm{C}\ell(X)$.

Proof. Show that any prime divisor $[Y]$ is Cartier. First: $U = X \setminus Y$, $[Y]|_U = 0$. Let $P \in Y$, then $I(Y) \cdot \mathscr{O}_{X,P} \subset \mathscr{O}_{X,P}$ is a height 1 prime, so $I(Y) \cdot \mathscr{O}_{X,P} = (f) \subset \mathscr{O}_{X,P}$, $f \in \mathscr{O}_{X,P} \subset k(X)$.

Note: $v_Y(f) = 1$, if $Z \neq Y$ prime divisor, $p \in Z$, then $f \in \mathscr{O}_{X,Z}$ (defined at P) and $f \notin I(Z) \cdot \mathscr{O}_{X,P}$. Thus, $v_Z(f) = 0$, and we have $(f) = [Y] + \sum n_i [Z_i]$ where $p \notin Z_i$ for all i.

Set $U = X \setminus (\cup Z_i)$ open in X, $p \in U$. Then $[Y]|_U = (f)|_U \in \mathrm{Div}(U)$ principal, so $[Y]$ is Cartier. □

Example: $X = V(xy - z^2) \subset \mathbb{A}^3$, $X_0 = X \setminus \{0,0,0\}$, X_0 is nonsingular, so $\mathrm{Pic}(X_0) = \mathrm{C}\ell(X_0) = \mathrm{C}\ell(X) = \mathbb{Z}/2\mathbb{Z}$, so there exists a unique nontrivial line bundle on X_0 which is NOT equal to the restriction of a line bundle on X.

Definition 5.13 (Affine, Finite)**.** *Let $f : X \to Y$ be a morphism of varieties. f is affine if $f^{-1}(V) \subset X$ is affine for all $V \subset Y$ is open affine.*

f is finite if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$-module.

Exercise: Enough that this is true for an open affine cover of Y.

Examples: X, Y affine, $f : X \to Y$ morphism is affine.

$X \subset Y$ closed, then the inclusion is finite.

Divisors on Non-Singular Curves

Recall that X nonsing complete curve implies that X is projective. $X \subset C_K$ open, $K = k(X)$, $X = C_K$.

Lemma 5.9. *Let X be a complete, nonsingular curve, then any nonconstant morphism $f : X \to Y$ is finite.*

Proof. WLOG: Y is a curve. Thus, $f^* : k(Y) \subset k(X)$ is a finite field extension. Take $V \subseteq Y$ open affine, $k[V] \subset k(Y)$. Set $A = \overline{k[V]} \subset k(X)$, then A is a finitely generated $k[V]$-module.

$U = \text{Spec}-m(A)$ a nonsingular curve, $k(U) = k(X) \Rightarrow$ we have diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \subset & & \downarrow \subset \\ U & \longrightarrow & V \end{array}$$

Claim: $f^{-1}(V) = U$. $x \in f^{-1}(V) \Rightarrow k[V] \subseteq \mathscr{O}_{X,x}$, so $A \subset \mathscr{O}_{X,x}$, thus $\mathscr{O}_{X,x} = A_P$ for some $P \subset A$ prime.

Thus, $x = P \in U = \text{Spec}-m(A)$. □

Definition 5.14 (Degree of f)**.** *Let $f : X \to Y$ be a finite, dominant morphism, then* $\deg(f) = [k(X) : k(Y)]$.

Pullback of Divisors on Curves

$f : X \to Y$ a finite morphism of nonsingular curves. $Q \in Y$, $\mathfrak{m}_Q = (t) \subseteq \mathscr{O}_{Y,Q}$. If $f(P) = Q$ then $f^* : \mathscr{O}_{Y,Q} \to \mathscr{O}_{X,P}$, $f^*t \in \mathfrak{m}_P$.

Definition 5.15. $f^* : \text{Div}(Y) \to \text{Div}(X) : [Q] \to \sum_{P \in f^{-1}(Q)} v_P(t)[P]$.

Alternatively, if $D \in \text{Div}(Y)$, set $V = Y - \text{Supp}(D)$, then $s = 1 \in \Gamma(V, \mathscr{L}(D))$. Note: $(s) = D$. Then $f^*s \in \Gamma(f^{-1}(V), f^*\mathscr{L}(D))$ is the pullback.

Exercise: $f^*D = (f^*s) \in \text{Div}(X)$.

Definition 5.16 (Torsion Free)**.** *Let R be a domain, M an R-module. M is torsion free if $\forall a \in R, x \in M$, then $ax = 0$ implies $a = 0$ or $x = 0$.*

Fact: Any f.g. torsion-free module over a PID is free.

Definition 5.17 (Degree of a Divisor)**.** *X a nonsingular curve, $D = \sum n_i[P_i] = \sum n_i P_i \in \text{Div}(X)$. Set $\deg(D) = \sum n_i$*

Warning: If X is not complete, then deg is not defined on $\text{C}\ell(X)$.

Proposition 5.10. *$f : X \to Y$ is a finite morphism of nonsingular curves, $D \in \text{Div}(Y)$. Then $\deg(f^*D) = \deg(f)\deg(D)$.*

Proof. ETS if $Q \in Y$ a point, then $\deg(f^*Q) = \deg(f)$. $V \subseteq Y$ open affine with $Q \in V$. Then $f^{-1}(V) = \text{Spec}-m(A) \subset X$, $A = \overline{k[V]} \subset k(X)$.

$Q \subset k[V]$ a max ideal, set $B = A_Q = (k[V] \setminus Q)^{-1}A$. A finitely generated $k[V]$-module implies B f.g. $k[V]_Q = \mathscr{O}_{Y,Q}$-module. $\mathscr{O}_{Y,Q}$ a DVR, B torsion free, so B is free $\mathscr{O}_{Y,Q}$-module.

$\text{rank}_{\mathscr{O}_{Y,Q}}(B) = \dim_{k(Y)} k(X) = \deg(f)$. $\mathfrak{m}_Q = (t) \subset \mathscr{O}_{Y,Q}$. $\mathscr{O}_{Y,Q}/t\mathscr{O}_{Y,Q} = k$. Thus, $\dim_k(B/tB) = \deg(f)$.

Note: points in $f^{-1}(Q)$ correspond to max ideals in $P \subset A$ such that $P \cap k[V] = Q$, which correspond to max ideals in $A_Q = B$.

Write $f^{-1}(Q) = \{P_1, \ldots, P_s\}$, $P_i \subseteq A$ max ideals, $B = \cap_{i=1}^s B_{P_i} \Rightarrow tB = \cap_{i=1}^s tB_{P_i} = \cap_{i=1}^s (tB_{P_i} \cap B)$.

By the Chinese Remainder Theorem, $B/tB \simeq \oplus_{i=1}^s B/(tB_{P_i} \cap B)$.

Injective: clear

Surjective: $t \in P_i$ for all i, B_{P_i} DVR, so $tB_{P_i} = (P_i B_{P_i})^{n_i}$, so $tB_{P_i} \cap B \subseteq P_i B$ and $tB_{P_i} \cap B \not\subseteq P_j B$ for $j \neq i$.

Thus, this is an isomorphism after $\otimes B_{P_i}$ (only the i^{th} summand survives).

Now: $B/(tB_{P_i} \cap B) = (B/(tB_{P_i} \cap B))_{P_i} = (B/tB)_{P_i} = B_{P_i}/tB_{P_i} = \mathscr{O}_{X,P_i}/(t)$. Thus $\dim_k B/(tB_{P_i} \cap B) = v_{P_i}(t)$.

Thus, $\deg(f^*Q) = \sum v_{P_i}(t) = \dim_k(B/tB) = \deg(f)$. □

Lemma 5.11. *$h \in k(Y)^*$ implies $f^*((h)) = (f^*h)$. $f^*h = h \circ f \in k(X)$.*

Proof. Let $P \in X$, $Q = f(P) \in Y$. $\mathfrak{m}_P = (s) \subseteq \mathscr{O}_{X,P}$. $\mathfrak{m}_Q = (t) \subset \mathscr{O}_{Y,Q}$.

$h = ut^m$, $u \in \mathscr{O}_{Y,Q}$ a unit. $f^*t = vs^n$, $v \in \mathscr{O}_{X,P}$ a unit.

Coef of $[P]$ in $f^*((h)) = v_Q(h)v_P(t) = mn$.

$f^*h = f^*(ut^m) = (f^*u)(f^*t)^m = (f^*u)v^m s^{nm}$, so coef of $[P]$ in (f^*h) is nm. □

So, we have a group homomorphism $f^* : \mathrm{C}\ell(Y) \to \mathrm{C}\ell(X)$.

Corollary 5.12. *X complete nonsing curve, $f \in k(X)^*$ implies* $\deg((f)) = 0$.

Proof. f is defined on open $U \subset X$. Then $f : U \to \mathbb{A}^1 \subset \mathbb{P}^1$ is a regular function. As $\mathbb{P}^1$ is complete, f extends to $f : X \to \mathbb{P}^1$. As X is complete, f is finite.

$k[\mathbb{A}^1] = k[t]$, $f^*(t) = f \in k(X)$. $(f) = (f^*t) = f^*((t))$, so $\deg((f)) = \deg(f^*((t))) = \deg(f)\deg(t)$.

$(t) = [0] - [\infty] \in \text{Div}(\mathbb{P}^1)$ so $\deg((t)) = 0$. □

So if X is a complete nonsingular curve, there exists $\deg : \mathrm{C}\ell(X) \to \mathbb{Z}$.

Notation: $D, D' \in \text{Div}(X)$, $D \sim D'$ iff $D = D' \in \mathrm{C}\ell(X)$.

Proposition 5.13. *If X complete nonsingular curve, then X rational iff $\exists P \neq Q \in X$ such that $P \sim Q$.*

Proof. $\Rightarrow$: $X = \mathbb{P}^1$, then $P \sim Q$ for all $P, Q \in \mathbb{P}^1$.

$\Leftarrow$: $\exists f \in k(X)^*$ such that $(f) = [P] - [Q] \in \text{Div}(X)$. $f : X \to \mathbb{P}^1$ a morphism. Then $(f) = (f^*t) = f^*((t)) = f^*([0] - [\infty])$.

This tells us that $f^*([0]) = [P]$ and $f^*([\infty]) = [Q]$. So $\deg(f^*[0]) = 1 = \deg(f) \cdot 1$, so f is degree 1, so it is birational. Thus isomorphism. □

Definition 5.18 (Elliptic Curve)**.** *An elliptic curve is a nonsingular closed plane curve* $E \subset \mathbb{P}^2$ *such that* $\deg(E) = 3$.

Example: $V_+(zy^2 - x^3 + z^2x) \subset \mathbb{P}^2$.

Claim: No elliptic curve is rational.

Exercise: Set $\mathscr{O}_E(1) = \mathscr{O}_{\mathbb{P}^2}(1)|_E$. Then $\Gamma(E, \mathscr{O}_E(1)) = (S/I(E))_1$, $S = k[x, y, z]$.

Therefore, $\dim_k \Gamma(E, \mathscr{O}_E(1)) = 3$.

Let $L \subset \mathbb{P}^2$ be a line. $L.E = \sum_{P' \in L \cap E} I(L \cdot E; P')[P'] = P + Q + R \in \mathrm{Div}(E)$.

Exercise: $\mathscr{O}_E(1) = \mathscr{O}_{\mathbb{P}^2}([L])|_E = \mathscr{O}_E([L.E])$, so $\exists D (= L.E)$, $D \in \mathrm{Div}(E)$ such that $\deg(D) = 3$ and $\dim_k \Gamma(E, \mathscr{O}_E(D)) = 3$.

Let $D \in \mathrm{Div}(\mathbb{P}^1)$ such that $\deg(D) = 3$. Then $\mathscr{O}_{\mathbb{P}^1}(D) = \mathscr{O}_{\mathbb{P}^1}(3)$, this gives us that $\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(D)) = \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(3)) = k[x_0, x_1]_3$. The dimension of this is 4.

Conclude: E is not rational.

Let $X \subset \mathbb{P}^2$ be any nonsingular curve. $\deg : \mathrm{C}\ell(X) \to \mathbb{Z}, [P] \mapsto 1$.

Definition 5.19. $\mathrm{C}\ell^0(C) = \ker(\deg)$. *So we have a short exact sequence* $0 \to \mathrm{C}\ell^0(X) \to \mathrm{C}\ell(X) \to \mathbb{Z} \to 0$ *that splits, so* $\mathrm{C}\ell(X) = \mathrm{C}\ell^0(X) \oplus \mathbb{Z}$.

Fact: $\mathrm{C}\ell^0$ corresponds to a nonginular complete abelian algebraic group, the Jacobi Variety of X.

Let $L = V_+(f)$, $M = V_+(g)$ be lines in $\mathbb{P}^2$.

$X.L = \sum_P I(X \cdot L; P)P = P_1 + \ldots + P_n$ where $n = \deg(X)$. $X.M = Q_1 + \ldots + Q_n$.

Exercise: $f/g \in k(X)^*$ and $(f/g) = X.L - X.M \in \mathrm{Div}(X)$.

$X.L - X.M = P_1 + P_2 + P_3 - Q_1 - Q_2 - Q_3 = 0 \in \mathrm{C}\ell(X)$ for $X = E$.

Theorem 5.14. *Let* $P_0 \in E$ *be any point, then* $E \to \mathrm{C}\ell^0(E)$ *by* $P \mapsto P - P_0$ *is bijective.*

Proof. Injective: If $P - P_0 = Q - P_0 \in \mathrm{C}\ell^0(E)$ then $P \sim Q$ so $P = Q$ as E is not rational.

Surjective: Let $M \subset \mathbb{P}^2$ be tangent line to E at P_0. $M.E = 2P_0 + R$, $R \in E$. Let $D \in \mathrm{C}\ell^0(E)$. Write $D = \sum n_i(Q_i - P_0)$ for $Q_i \in E$, $n_i \in \mathbb{Z}$.

Assume that $n_i < 0$. Then L =line through Q_i and R, $L.E = Q_i + R + Q_i'$, $0 = L.E - M.E = Q_i + R + Q_i' - 2P_0 - R$ so $Q_i - P_0 = -(Q_i' - P_0) \in \mathrm{C}\ell^0(E)$.

Replace Q_i by Q_i', $n_i \mapsto -n_i$, WLOG, $n_i \geq 0$.

Claim: $D = P - P_0 \in \mathrm{C}\ell^0(E)$, $P \in E$. Induction on $\sum n_i$.

If $\sum n_i \geq 2$, then $Q_1 - P_0$, $Q_2 - P_0$ have positive coefficients. L =line through Q_1, Q_2, $L.E = Q_1 + Q_2 + Q' \in \mathrm{Div}(E)$.

Let L' be the line through Q' and P_0. Then $L'.E = Q' + P_0 + Q''$. $L.E - L'.E = Q_1 + Q_2 - P_0 - Q'' = 0$, so $(Q_1 - P_0) + (Q_2 - P_0) = (Q'' - P_0) \in \mathrm{C}\ell^0(E)$. □

Example: $\mathrm{char}\, k \neq 2$, $\lambda \in k$, $\lambda \neq 0, 1$. $E_\lambda = V_+(zy^2 - x(x - z)(x - \lambda z)) \subset \mathbb{P}^2$. Take $P_0 = (0 : 1 : 0) \in E$ E_λ corresponds to $\mathrm{C}\ell^0(E_\lambda)$ by $P \mapsto P - P_0$. Let $\oplus$ be a group op on W. $Q_1 \oplus Q_2 = Q''$ (Picture omitted)

Fact: Any elliptic curve is isomorphic to E_λ by $P_0 \leftrightarrow (0 : 1 : 0)$.

Theorem 5.15. *E is an algebraic group.*

Proof. (char $k \neq 2$): WLOG, $E = E_\lambda$, $P_0 = (0:1:0)$. Define $\varphi : E \times E \to E$ by $\varphi(P,Q) = R$ the unique point such that $\exists$ a line L with $L.E = P+Q+R$. It is enough to show that $\varphi : E \times E \to E$ is a morphism.

$P \oplus Q = \varphi(P_0, \varphi(P,Q))$, $-P = \varphi(P_1, P_0)$. Set $U_1 = D_+(z)$ and $U_2 = D_+(y)$ subsets of E. $E = U_1 \cup U_2$.

Show that $(U_i \times U_j) \cap \varphi^{-1}(U_\ell) \to U_\ell$ by φ is a morphism for all $i,j,\ell \in \{0,1\}$.

$U_1 = V(y^2 - x(x-1)(x-\lambda)) \subset \mathbb{A}^2$. $(U_1 \times U_1) \cap \varphi^{-1}(U_1) \to U_1 \to k$ is the regular function $(x_1,y_1)\times(x_2,y_2) \mapsto \begin{cases} \frac{(y_2-y_1)^2}{(x_2-x_1)^2} - (x_1+x_2) + 1 + \lambda & x_1 \neq x_2 \\ \left(\frac{x_1^2+x_1x_2+x_2^2+\lambda-(1+\lambda)(x_1+x_2)}{y_1+y_2}\right)^2 + 1 + \lambda - (x_1+x_2) & y_1+y_2 \neq 0 \end{cases}$ □

Differentials

R is a ring, S is a commutative R-algebra, M an S-module.

Definition 5.20 (R-derivation)**.** *A function $D : S \to M$ is an R-derivation if $D(fg) = fD(g) + gD(f)$ for all $f,g \in S$, $D(f+g) = D(f) + D(g)$, and $D(f) = 0$ for all $f \in R$.*

Remark: the third conditioon holds iff D is an homomorphism of R-modules. $\Rightarrow$: $f \in R \Rightarrow D(fg) = fD(g)$ and $\Leftarrow$ is an exercise (use $D(1) = 0$).

Definition 5.21 (Module of Kähler differentials)**.** *F=free S-module with basis $\{d(f) | f \in S\} = \oplus_{f \in S} S \cdot d(f)$. F' =submodule generated by $d(f)$ for $f \in R$, $d(fg) - fd(g) - gd(f)$, $d(f+g) - d(f) - d(g)$.*

We define $\Omega_{S/R} = F/F'$ is the module of Kähler differentials of S over R

We define $d = d_S = d_{S/R} : S \to \Omega_{S/R}$ by $f \mapsto d(f) + F'$. This is the universal R-derivation of S.

It has the universal property that given any R-derivation $D : S \to M$, there exists a unique map S-homomorphism $\tilde{D} : \Omega_{S/R} \to M$ such that $D = \tilde{D} \circ d_S$.

Exercise: Let $P(x_1,\dots,x_n) \in R[x_1,\dots,x_n]$ and $f_1,\dots,f_n \in S$, and $D : S \to M$ is an R-derivation. Then $D(P(f_1,\dots,f_n)) = \sum_{i=1}^n \frac{\partial P}{\partial x_i}(f_1,\dots,f_n) D(f_i)$.

Consequence: If S generated by $f_1,\dots,f_n$ as an R-algebra, then $\Omega_{S/R}$ is gererated by $d_S(f_1),\dots,d_S(f_n)$ as an S-module.

Proposition 5.16. $S = R[x_1,\dots,x_n]$. *Then $\Omega_{S/R}$ is the free S-module on $dx_1,\dots,dx_n$.*

Proof. Have a surjectve S-hom from $S^n \to \Omega_{S/R}$ which sends $e_i \mapsto dx_i$. This is surjective. We define $D : S \to S^n$ by $P(x_1,\dots,x_n) \mapsto \left(\frac{\partial P}{\partial x_1},\dots,\frac{\partial P}{\partial x_n}\right)$. By the universal property, there is a unique S-homomorphism $\tilde{D} : \Omega_{S/R} \to S^n$, by definition, $d(x_i) \mapsto D(x_i) = e_i$, so this is an inverse. □

Proposition 5.17. *Given ring homomorphisms $R \to S \to Y$ then we have an exact sequence of T-modules $\Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to \Omega_{T/S} \to 0$.*

Proof. $S \to T \xrightarrow{d_T} \Omega_{T/R}$ is an R-deriv of S. So we get S-hom $\varphi : \Omega_{S/R} \to \Omega_{T/R}$ via $\varphi(d_S(f)) = d_T(f)$. Thus, we have a T-hom $\Omega_{S/R} \otimes T \overset{\tilde{\varphi}\to}{\Omega}_{T/R}$ by $\omega \otimes h \mapsto h\varphi(\omega)$.

Note: Image($\tilde{\varphi}$) =submodule of $\Omega_{T/R}$ generated by $d_T(f)$ for $f \in S$. Thus $\Omega_{T/R}/\operatorname{Im}(\tilde{\varphi}) = \Omega_{T/S}$. □

Note: $I \subset S$ an ideal, $T = S/I$, then I/I^2 is a T-module, $T \times I/I^2 \to I/I^2$ by $(f + I) \cdot (h + I^2) = fh + I^2 \in I/I^2$.

Proposition 5.18. *$T = S/I$. We have an exact sequence of T-modules $I/I^2 \to \Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to 0$, where the first map is given by $h + I^2 \mapsto d_s(h) \otimes 1$.*

Proof. Set M equal to the image of I/I^2 in $\Omega_{S/R} \otimes_S T$. Then M is generated by $\{d_s(h) \otimes 1 | h \in I\}$.

We define $D : T \to (\Omega_{S/R} \otimes T)/M$ by $D(f + I) = (d_S(f) \otimes 1) + M$. This is an R-derivation.

Thus, there is a unique T-hom $\tilde{D} : \Omega_{T/R} \to (\Omega_{S/R} \otimes_S T)/M$ by $d_T(f + I) \mapsto (d_s(f) \otimes 1) + M$ □

Example: $S = R[x_1, \ldots, x_n]$, $I = (f_1, \ldots, f_p) \subset S$. $T = S/I$. $\Omega_{S/R} \otimes_S T = \oplus_{i=1}^n T dx_i = T^{\oplus n}$.

The image of f_i under $I/I^2 \to T^{\oplus n}$: $d_s(f_i) \otimes 1 = \left(\frac{\partial \bar{f}_i}{\partial x_1}, \ldots, \frac{\partial \bar{f}_i}{\partial x_n} \right)$. Set $J = \left(\frac{\partial \bar{f}_i}{\partial x_j} \right) \in Mat(p \times n; T)$ is the Jacobi matrix.

So the image of f_i is $e_i J$, so $\Omega_{T/R} = \operatorname{coker}(I/I^2 \to \Omega_{S/R} \otimes T) = \operatorname{coker}(T^p \xrightarrow{J} T^n)$.

e.g. $T = k[x, y]/(y^2 - x^3 + x)$, so $J = [1 - 3x^2, 2y]$, so $\Omega_{T/k} = T \oplus T/\langle (1 - 3x^2)e_1 + (2y)e_2 \rangle$.

Proposition 5.19. *S an R-algebra, $U \subseteq S$ multiplicatively closed subset, then $\Omega_{U^{-1}S/R} = U^{-1}\Omega_{S/R}$*

Proof. $S \to U^{-1}S \to \Omega_{U^{-1}S/R}$ is an R-derivation. Thus, it induces an S-homomorphism $\Omega_{S/R} \to \Omega_{U^{-1}S/R}$, $d_S(f) \mapsto d(f)$, where d is the universal derivation of $U^{-1}S$.

This induces $U^{-1}S$-hom $U^{-1}\Omega_{S/R} \to \Omega_{U^{-1}S/R}$ by $d_s(f)/u \mapsto u^{-1}d(f)$.

We define $D : U^{-1}S \to U^{-1}\Omega_{S/R}$ by $D(s/u) \mapsto \frac{ud(s) - sd(u)}{u^2}$. Exercise: D is well defined R-derivation.

This induces $\tilde{D} : \Omega_{U^{-1}S/R} \to U^{-1}\Omega_{S/R}$ is the inverse map. □

Let X be a topological space. $\mathscr{R}, \mathscr{S}$ sheaves of rings on X, $\mathscr{R} \to \mathscr{S}$ a ring hom.

Definition 5.22. *$pre - \Omega_{\mathscr{S}/\mathscr{R}}(U) = \Omega_{\mathscr{S}(U)/\mathscr{T}(U)}$ for $U \subseteq X$ open. For $V \subset U$ open, $\mathscr{S}(U) \to \mathscr{S}(V) \xrightarrow{d} pre - \Omega(V)$ is an $R(U)$-derivation. So, we get $\mathscr{S}(U)$-hom $pre - \Omega(U) \to pre - \Omega(V)$.*

We define $\Omega_{\mathscr{S}/\mathscr{R}} = (pre - \Omega_{\mathscr{S}/\mathscr{R}})^+$, the sheafification.

Let $\varphi : X \to Y$ morphism of varieties, then we have ring hom $\varphi^* : \varphi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$.

Definition 5.23 (Relative cotangent sheaf). *$\Omega_{X/Y} = \Omega_{\mathscr{O}_X/\varphi^{-1}\mathscr{O}_Y}$ is called the relative cotangent sheaf*

Special case: $X \to \{pt\}$, $\Omega_X = \Omega_{X/k} = \Omega_{X/\{pt\}}$. This is called the cotangent sheaf.

Proposition 5.20. *$\varphi : X \to Y$ a morphism of affine varieties, then $\Omega_{X/Y} = \widetilde{\Omega_{k[X]/k[Y]}}$*

Proof next time.

As a consequence, $\Omega_{X/Y}$ is always coherent.

Lemma 5.21. *If $(A, \mathfrak{m})$ is a local Nötherian domain, N a finitely generated A-module, then we set $r = \dim_{A/\mathfrak{m}}(N/\mathfrak{m}N)$. If $r \leq \dim_{A_0}(N_0)$, then N is free of rank r.*

Proof. Nakayama's Lemma implies that N can be generated by r elements. Thus, there exists an exact sequence $0 \to K \to A^r \to N \to 0$, localization is exact, so $0 \to K_0 \to A_0^r \to N_0 \to 0$ is exact, so the last morphism is an isomorphism of vector spaces, so $A_0^r \simeq N_0$, so $K_0 = 0$. Thus, $K = 0$ as it is torsion free and localizes to zero, so $A^r \simeq N$. $\square$

Recall: Let $X \subset \mathbb{A}^n$ be a closed irreducible variety. Let $I = I(X) = (f_1, \ldots, f_s)$. Let $P \in X$. Set $M = I(\{p\}) \subset k[\mathbb{A}^n]$, $M/M^2 \simeq k^n$ via $h + M^2 \mapsto \left(\frac{\partial h}{\partial x_1}(P), \ldots, \frac{\partial h}{\partial x_n}(P)\right)$

So we have an exact sequence $(I + M^2)/M^2 \to M/M^2 \to \mathfrak{m}_P/\mathfrak{m}_P^2 \to 0$. $\mathfrak{m}_P \subset \mathscr{O}_{X,P}$ a max ideal, therefore $k^s \to k^n \to \mathfrak{m}_P/\mathfrak{m}_P^2 \to 0$ is also exact where the first map is $J(P)$, and we call the second ϕ.

If $h \in M$, then $\phi\left(\frac{\partial h}{\partial x_1}(P), \ldots, \frac{\partial h}{\partial x_n}(P)\right) = h + \mathfrak{m}_P^2$.

Note: $\operatorname{rank}(k(X)^s \xrightarrow{J} k(X)^n) \leq c = \operatorname{codim}(X; \mathbb{A}^n)$. (If h is any $(c+1) \times (c+1)$-minor of J, then $h \in k[X]$, and $h(P) = (c+1) \times (c+1)$-minor. $J(P) = 0$ for all $P \in X$. So $h = 0 \in k[X]$.)

Theorem 5.22. *Assume X is an irreducible variety of dimension r, let $P \in X$. Then P is a nonsingular point iff $\Omega_{X,P} \simeq \mathscr{O}_{X,P}^{\oplus r}$. If $\mathfrak{m}_P$ is generated by $h_1, \ldots, h_r \in \mathfrak{m}_P$, then $dh_1, \ldots, dh_r \in \Omega_{X,P}$ is a basis for $\Omega_{X,P}$.*

Proof. WLOG: $X \subseteq \mathbb{A}^n$ affine. $I(X) = (f_1, \ldots, f_s)$, $J = \left(\frac{\partial f_i}{\partial x_j}\right)$.

$k[X]^s \xrightarrow{J} k[X]^n \to \Omega_{k[X]/k} \to 0$ yields $\mathscr{O}_{X,P}^s \xrightarrow{J} \mathscr{O}_{X,P}^n \to \Omega_{X,P} \to 0$, which we will call $(*)$.

We mod out by $\mathfrak{m}_P$, and get $k^s \xrightarrow{J(P)} k^n \to \Omega_{X,P}/\mathfrak{m}_P\Omega_{X,P} \to 0$.

Thus, $\mathfrak{m}_P/\mathfrak{m}_P^2 \simeq \Omega_{X,P}/\mathfrak{m}_P\Omega_{X,P}$ by $h + \mathfrak{m}_P^2 \mapsto \sum_{j=1}^n \frac{\partial h}{\partial x_j}(P)dx_j$.

Assume that $\Omega_{X,P}$ is free of rank r, then $\dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = r$, thus P is a nonsingular point.

Assume that $\dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = r$. $(*) \Rightarrow k(X)^s \xrightarrow{J} k(X)^n \to (\Omega_{X,P})_0 \to 0$ is exact.

Note: $r = \dim_k(\Omega_{X,P}/\mathfrak{m}_P\Omega_{X,P}) \leq \dim_{k(X)}((\Omega_{X,P})_0)$.

The lemma implies that $\Omega_{X,P} \simeq \mathscr{O}_{X,P}^{\oplus r}$. □

Lemma 5.23. *$\varphi : X \to Y$ is a morphism of affine varieties. Then $\Gamma(X, pre - \Omega_{X/Y}) = \Omega_{k[X]/k[Y]}$*

Proof. $S = \Gamma(X, \varphi^{-1}\mathscr{O}_Y)$, ring homomorphisms $k[Y] \to S \to k[X]$. Thus, $\Omega_{S/k[Y]} \otimes_S k[X] \to \Omega_{k[X]/k[Y]} \to \Omega_{k[X]/S} \to 0$ where the last is $\Gamma(X, pre - \Omega_{X/Y})$, so enough to show that the first map is zero.

Let $f \in \mathrm{Im}(S \to k[X])$. We must show that $df = 0 \in \Omega_{k[X]/k[Y]}$. There exists open cover $X = \cup_{i=1}^n U_i$ such that $f|_{U_i} \in$ image of $\Gamma(U_i, pre - \varphi^{-1}\mathscr{O}_Y) = \varinjlim_{V \supset \varphi(U_i)} \mathscr{O}_Y(V)$

WLOG, $U_i = X_{g_i}$ where $g_i \in k[X]$. Enough to show that $df = 0$ in $\left(\Omega_{k[X]/k[Y]}\right)_{g_i}$ for each i, since $g_i^N df = 0 \in \Omega_{k[X]/k[Y]}$ and $(g_i^N, \dots, g_n^N) = (1) = k[X]$.

But $\left(\Omega_{k[X]/k[Y]}\right)_{g_i} = \Omega_{k[U_i]/k[Y]}$. So we replace X with U_i, we may assume that $f \in$ image of $\Gamma(X, pre - \varphi^{-1}\mathscr{O}_Y) = \varinjlim_{V \supset \varphi(X)} \mathscr{O}_Y(V)$. I.E. there exists $V \subset Y$ open, $f' \in \mathscr{O}_Y(V)$ such that $\varphi(X) \subset V$ and $f = \varphi^*(f') \in k[X]$. Now $V = \cup_{i=1}^m Y_{h_i}$, $h_i \in k[Y]$, $f' \in k[Y]_{h_i}$, so $h_i^N f' \in k[Y]$ for all i, so $h_i^{N+1} df = d(h_i^{N+1} f) = 0 \in \Omega_{k[X]/k[Y]}$.

Now, $X = \cup X_{h_i} \Rightarrow (h_1^N, \dots, h_m^N) = (1) \subset k[X]$, so $df = 0$. □

Proposition 5.24. *$\varphi : X \to Y$ morphism of affines. Then $\Omega_{X/Y} = \widetilde{\Omega_{k[X]/k[Y]}}$*

Proof. Set $\Omega = \Omega_{k[X]/k[Y]}$. We have $\Omega \simeq \Gamma(X, pre - \Omega_{X/Y}) \to \Gamma(X, \Omega_{X/Y})$, this gives an $\mathscr{O}_X$-homomorphism $\tilde{\Omega} \to \Omega_{X/Y}$.

Let $f \in k[X]$. $\Gamma(X_f, \tilde{\Omega}) = \Omega_f = \Omega_{k[X_f]/k[Y]} = \Gamma(X_f, pre - \Omega_{X/Y})$.

$\{X_f\}$ is a basis for the top, and so they have the same stalks. □

Corollary 5.25. *$\varphi : X \to Y$ any morphism of varieties. Then $\Omega_{X/Y}$ is coherent.*

Proof. Let $Y = \cup V_i$ open affine cover. $\varphi^{-1}(V_i) = \cup U_{ij} \subseteq X$ is an open affine cover of X. $\Omega_{X/Y}|_{U_{ij}} = \tilde{\Omega}_{k[U_{ij}]/k[V_i]}$. □

Corollary 5.26. *If X irreducible, then X is nonsingular iff Ω_X is a locally free $\mathscr{O}_X$-module.*

Example: $X = \mathbb{P}^1$, $\Omega_{\mathbb{P}^1}$ is a line bundle. The projective coordinate ring is $k[x_0, x_1]$. Set $t = \frac{x_1}{x_0} \in k(\mathbb{P}^1)^*$. $t \in \mathscr{O}_{\mathbb{P}^1}(D_+(x_0))$, $dt \in \Gamma(D_+(x_0), \Omega_{\mathbb{P}^1})$. Find $(dt) \in \mathrm{Div}(\mathbb{P}^1)$.

$U_i = D_+(x_i)$, $U_0 = \mathbb{A}^1 \subset \mathbb{P}^1$. If $p \in U_0$, then $t - p$ generated $\mathfrak{m}_p$, so $\Omega_{\mathbb{P}^1,p}$ is generated by $d(t-p) = dt$, so $v_p(dt) = 0$ for all $p \in U_0$.

$k[U_1] = k[s]$, $s = t^{-1}$, $dt = d(s^{-1}) = -s^{-2}ds$, $v_\infty(dt) = v_\infty(s^{-2}) = -2$. Thus $(dt) = -2[\infty] \in \mathrm{Div}(\mathbb{P}^1)$, and so $\Omega_{\mathbb{P}^1} \simeq \mathscr{O}_{\mathbb{P}^1}(-2[pt]) = \mathscr{O}_{\mathbb{P}^1}(-2)$.

Example: $E = E_\lambda \subset \mathbb{P}^2$ an elliptic curve. Then $\Omega_E \simeq \mathscr{O}_E$.

Linear Systems

Let X be a nonsingular projective variety. $D = \sum n_Y[Y] \in \mathrm{Div}(X)$. We say that D is effective if $n_Y \geq 0$ for all Y. If D is effective, we write $D \geq 0$.

Definition 5.24 (Complete Linear System of D)**.** *Given any $D \in \mathrm{Div}(X)$, define $|D| = \{D' \in \mathrm{Div}(X) : D' \sim D$ and $D' \geq 0\}$.*

Theorem 5.27. *If X is projective and $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, then* $\dim_k \Gamma(X, \mathscr{F}) < \infty$.

Definition 5.25. *Let $\ell(D) = \dim_k \Gamma(X, \mathscr{O}_X(D))$.*

Theorem 5.28. $\mathbb{P}(\Gamma(X, \mathscr{O}_X(D))) \to |D|$ *by $s \mapsto (s)$ is bijective.*

Proof. Let $s \in \Gamma(X, \mathscr{O}_X(D))$, then $(s) \geq 0$ and $(s) \sim D$. So the map is well defined.

Injective: Suppose $s_1, s_2 \in \Gamma(X, \mathscr{O}_X(D))$, assume that $(s_1) = (s_2) \in \mathrm{Div}(X)$. Then $(s_1/s_2) = (s_1) - (s_2) = 0$, so $s_1/s_2 \in k[X] = k$.

Surjective: Let $D' \in |D|$. Then $D' \sim D$, so $D' = D + (f)$ where $f \in k(X)^*$. We define s to be the section given by $f \in \Gamma(X, \mathscr{O}_X(D))$. This is a global section, because $v_Y(f) \geq -n_Y$ for all Y, ($D = \sum n_Y[Y]$). Set $s_0 = 1 \in \Gamma(U, \mathscr{O}_X(D))$. $(s) = (f \cdot s_0) = (f) + (s_0) = (f) + D = D'$. ☐

Lemma 5.29. *X is a complete nonsingular curve, $D \in \mathrm{Div}(X)$, if $\ell(D) \neq 0$ then $\deg(D) \geq 0$ and if $\ell(D) \neq 0$ and $\deg(D) = 0$ then $D \sim 0$.*

Proof. $\ell(D) \neq 0$, then $|D| \neq \emptyset$, so $D \sim D' \geq 0$. So $\deg(D) = \deg(D') \geq 0$.

If $\deg(D) = 0$, then $\deg(D') = 0$, but as D' is effective, $D' = 0$. ☐

Riemann-Roch Theorem

Let X be a complete nonsingular curve.

Definition 5.26 (Canonical Divisor)**.** *$K \in \mathrm{Div}(X)$ is a canonical divisor on X if $\Omega_X \simeq \mathscr{O}(K)$.*

Definition 5.27 (Genus)**.** *The genus $g = \ell(K) = \dim_k \Gamma(X, \Omega_X)$*

Example: $X = \mathbb{P}^1$, $\Omega_{\mathbb{P}^1} = \mathscr{O}(-2)$, so $g = 0$.

Example: $E = E_\lambda \subset \mathbb{P}^2$ elliptic curve, $\Omega_E \simeq \mathscr{O}_E$. So $g = 1$.

Theorem 5.30 (Riemann-Roch)**.** *For any $D \in \mathrm{Div}(X)$ where X is a complete nonsingular curve, we have $\ell(D) + \ell(K - D) = \deg(D) + 1 - g$.*

Example: $X = \mathbb{P}^1$, $K = -2P$ for some $P \in \mathbb{P}^1$. The RRT theorem says that $\ell(nP) + \ell(-2P - nP) = n + 1 - 0$, so if $n \geq 0$, we have that $\ell(nP) = n + 1$. If $n = -1$, then $0 + 0 = -1 + 1 = 0$. If $n \geq -2$, we also see that it works.

Example: Set $D = K$, then $\ell(K) - \ell(K - K) = \deg(K) + 1 - g$, so $g - 1 = \deg(K) + 1 - g$, so $\deg(K) = 2g - 2$.

Corollary 5.31. *A nonsingular curve is either affine or projective.*

Proof. C nonsingular curve implies that $C = C_K \setminus \{P_1, \dots, P_n\}$ where $K = k(C)$, $X = C_K$. If $m >> 0$, then $\ell(mP_i) = m+1-g \geq 2$. $1, f_i \in \Gamma(X, \mathscr{O}_X(mP_i))$, $f_i \notin k$.

$(f_i) = -r_i[P_i]$ effective divisor in $\mathrm{Div}(X)$. Set $f = \sum_{i=1}^n f_i \in k(X)^*$. f is defined exactly on $C \subsetneq X$, so $C \simeq \mathrm{Spec}-m(\overline{k[f]})$ is affine. □

Corollary 5.32. *X is rational iff $g = 0$.*

Proof. $\Rightarrow$: genus of $\mathbb{P}^1$ is 0.

$\Leftarrow$: Let $P \neq Q \in X$. Set $D = P - Q \in \mathrm{Div}(X)$. By Riemann-Roch, $\ell(D) \geq \deg(D) + 1 - g$, so $\ell(D) \geq 1$. Thus $|D| \neq \emptyset$, so there is $D' \geq 0$ such that $D \sim D'$, but $\deg(D') = 0$, so $D' = 0$. □

Corollary 5.33. *X complete nonsingular curve of $g = 1$, $P_0 \in X$, $\mathrm{char}(k) \neq 2$. Then $\exists$ isomorphism $X \simeq E_\lambda = V_+(zy^2 - x(x-z)(x-\lambda z)) \subset \mathbb{P}^2$ for some λ not 0 or 1, that sends $P_0 \mapsto (0:1:0)$.*

Proof. $\deg(K) = 2g - 2 = 0$, so Riemann-Roch implies $\ell(nP_0) = n + 1 - 1 = n$ for all $n \geq 1$, $k = \Gamma(\mathscr{O}_X) = \Gamma(\mathscr{O}_X(P_0)) \subsetneq \Gamma(\mathscr{O}_X(2P_0)) \subsetneq \dots \subsetneq k(X)$.

Take $x \in \Gamma(\mathscr{O}_X(2P_0)) \setminus k$. $v_{P_0}(x) = -2$ $(x) = A + B - 2P_0$ for $A, B \in X$.

$x : X \to \mathbb{P}^1$ is a morphism, $x^*([0] - [\infty]) = A + B - 2P_0$, so $x^*([0]) = A + B$, thus $[k(X) : k(x)] = \deg(x) = 2$.

Take $y \in \Gamma(\mathscr{O}_X(3P_0)) \setminus \Gamma(\mathscr{O}_X(2P_0))$. $v_{P_0}(y) = -3$, but as this is odd, $y \notin k(x)$. Thus $k(X) = k(x, y)$.

$\{1, x, y, x^2, xy\}$ is a basis for $\Gamma(\mathscr{O}_X(5P_0))$, $1, x, y, x^2, xy, x^3, y^2 \in \Gamma(\mathscr{O}_X(6P_0))$, $\dim = 6$.

So there exists a linear relations. Rescale x, y: $y^2 + b_1xy + b_0y = x^3 + a_2x^2 + a_1x + a_0$. Replace y with $y + \frac{1}{2}(b_1x + b_0)$, $y^2 = (x-a)(x-b)(x-c)$ where $a, b, c \in k$.

Claim: $a \neq b \neq c \neq a$.

$\varphi : X \setminus \{P_0\} \to C = V(y^2 - (x-a)(x-b)(x-c)) \subset \mathbb{A}^2$, $P \mapsto (x(P), y(P))$. This is birational, as $k(X) = k(x, y)$.

Assume $a = b = c$. Then C is a curve with a cusp, and is rational, so X would be rational, contradiction.

Assume $a = b \neq c$, then C is the nodal curve, which is also rational.

Replace x with $\frac{x-a}{b-a}$, rescale y, $y^2 - x(x-1)(x-\lambda)$ where $\lambda = \frac{c-a}{b-a} \neq 0, 1$.

$X \setminus \{P_0\} \to C \subset \mathbb{A}^2$ extends to an isomorphism $X \to E_\lambda$. □

Lemma 5.34. *X complete nonsingular curve of genus g. Let $P_0, Q_0, \dots, Q_g \in X$, then there exists $P_1, \dots, P_g \in X$ such that $\sum_{i=0}^g P_i \sim \sum_{i=0}^g Q_i$.*

Proof. WLOG $P_0 \neq Q_i$ for all i.

Set $D = \sum Q_i$. $\ell(D) \geq \deg(D) + 1 - g = 2$. Thus, $\exists h \in \Gamma(X, \mathscr{O}_X(D))$ such that $h \notin k$. Set $f = h - h(P_0) \in k(X)^*$. $(f) = -D + P_0 + P_1 + \dots + P_g = 0 \in \mathrm{C}\ell(X)$. □

Corollary 5.35. *The Map $X^g = X \times \dots \times X$ with g factors to $\mathrm{C}\ell^0(X)$ by $(P_1, \dots, P_g) \mapsto \sum_{i=1}^g (P_i - P_0)$ is surjective.*

Proof. Note: Let $Q \in X$, the lemma implies that there are $P_1, \ldots, P_g \in X$ such that $(g+1)P_0 \sim Q + P_1 + \ldots + P_g$, so $-(Q - P_0) = \sum_{i=1}^{g}(P_i - P_0) \in \mathrm{C}\ell^0(X)$.

Let $D = \sum n_Q(Q - P_0) \in \mathrm{C}\ell^0(X)$. The note implies that we can assume $n_Q \geq 0$, and the lemma implies that we may assume $\sum n_Q \leq g$. □

Blow-Up of Varieties

Y affine variety, $X \subseteq Y$ closed, $I = I(X) = (f_0, \ldots, f_n) \subset k[Y]$. Let $\varphi : Y \setminus X \to \mathbb{P}^n$ by $\varphi(y) = (f_0(y) : \ldots : f_n(y))$.

Definition 5.28 (Blowup of Y along X.)**.** $B\ell_X(Y) = \overline{(y, \varphi(y)) : y \in Y \setminus X} \subset Y \times \mathbb{P}^n$.

Note: If $Y \setminus X \subseteq Y$ is dense, then $\pi : B\ell_X(Y) \to Y$ is surjective because $\mathbb{P}^n$ is complete. And $\pi : \pi^{-1}(Y \setminus X) \to Y \setminus X$ is an isomorphism.

Point: If Y is singular along X, then usually $B\ell_X(Y)$ is less singular.

Example: $Y = V(y^2 - x^2(x+1)) \subset \mathbb{A}^2$. $I(X) = (x, y) \subset k[Y]$. $\varphi : Y \setminus \{0\} \to \mathbb{P}^1$, $P \mapsto$ line through O and P.

$B\ell_X(Y) = \{(P, \varphi(P)), (0, (1 : -1)), (0, (1 : 1))\}$.

Note: $\pi^{-1}(X) \subset B\ell_X(Y)$ is an effective Cartier divisor. i.e. codim = 1 and the ideal of $\pi^{-1}(X)$ is locally generated by a single element.

In fact: $\mathscr{L} = \pi^*_{\mathbb{P}^n}(\mathscr{O}_{\mathbb{P}^n}(1))$. Define $s_i = \pi^*_{\mathbb{P}^n}(z_i) \in \Gamma(B\ell_X(Y), \mathscr{L})$, $z_i \in \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$ for $0 \leq i \leq n$. We define $s = \sum_{i=0}^{n} f_i s_i \in \Gamma(B\ell_X(Y), \mathscr{L})$. Then $\pi^{-1}(X) = Z(s) \subset B\ell_X(Y)$.

Note: $B\ell_X(Y)$ is independent of the generators f_i of $I(X)$. $B\ell_X(Y) \subset Y \times \mathbb{P}^n$, set $J = I(B\ell_X(Y)) \subset k[Y][z_0, \ldots, z_n]$, a graded ideal.

It is a fact that one can recover $B\ell_X(Y)$ from $k[Y][z_0, \ldots, z_n]/J$.

Claim: $k[Y][z_0, \ldots, z_n]/J \simeq \oplus_{d \geq 0} I^d$ and this is by definition $\oplus I^d t^d \subset k[Y][t]$ and it is the subring of $k[Y][t]$ generated by $k[Y]$ as well as $tf_0, \ldots, tf_n$.

Morphism $\psi : Y \times \mathbb{A}^1 \to Y \times \mathbb{A}^{n+1}$ by $\psi(y, t) \mapsto (y, (tf_0(y), \ldots, tf_n(y)))$. $\psi(Y \times \mathbb{A}^1)$ =cone over $B\ell_X(Y)$, so $J = I(\psi(Y \times \mathbb{A}^1)) \subset k[Y][z_0, \ldots, z_n]$. $\psi^* : k[Y][z_0, \ldots, z_n] \to k[Y][t]$ with $z_i \mapsto tf_i$.

$J = \ker(\psi^*)$, so $k[Y][z_0, \ldots, z_n]/J \simeq \mathrm{Im}(\psi^*) = k[Y][tf_0, \ldots, tf_n] = \oplus_{d \geq 0} I^d$.

Now let Y be any variety, $X \subset Y$ any closed subset. Take and open affine cover $Y = \cup Y_i$. $B\ell_{Y_i \cap X}(Y_i)$ can be glued together to get $B\ell_X(Y) = \cup_i B\ell_{Y_i \cap X}(Y_i)$.

$\mathscr{I}_X \subset \mathscr{O}_Y$ a sheaf of ideals. $\oplus_{d \geq 0} \mathscr{I}_X^d$ is a sheaf of graded $\mathscr{O}_Y$-algebras, which can be turned into a variety, $B\ell_X(Y)$ (see next semester).

6 Schemes

We will try to state definitions and theorems from commutative algebra, but will not prove many of them, as our focus is geometry.

Let A be a ring

Definition 6.1 (Spec A)**.** *We define* $\operatorname{Spec} A = \{$*prime ideals* $P \subset A\}$.

If $I \subseteq A$, we set $V(I) = \{P \in \operatorname{Spec} A : I \subseteq P\}$.

Lemma 6.1. *1.* $V(IJ) = V(I \cap J) = V(I) \cup V(J)$

2. $V(\sum I_\alpha) = \cap V(I_\alpha)$

3. $V(0) = \operatorname{Spec} A$, $V(A) = \emptyset$

4. $V(I) \subseteq V(J) \iff \sqrt{I} \supseteq \sqrt{J}$

Proof. We begin with 4: This follows from the fact that $V(I) = V(\sqrt{I})$ and $\sqrt{I} = \cap_{P \in V(I)} P$.

1: $IJ \subset I \cap J \subseteq I$, so $V(IJ) \supseteq V(I \cap J) \supseteq V(I) \cup V(J)$. Let $P \in V(IJ)$. Assume it is not in $V(I)$. Then there exists $a \in I$ with $a \notin P$. Then for any $b \in J$, $ab \in IJ \subseteq P$, so $b \in P$, thus $J \subset P$, so $P \in V(J)$.

2 and 3 are straightforward exercises. □

Topology: Let $U \subset \operatorname{Spec}(A)$ be open iff $\operatorname{Spec}(A) \setminus U = V(I)$ for some $I \subset A$.

We note that $P \in \operatorname{Spec}(A)$, then $\overline{\{P\}} = V(P)$, so P is a closed point iff P is maximal.

Example: Take $k = \bar{k}$ an alg closed field, $A = k[x, y]$. Then $\operatorname{Spec}(A) = \{(x-a, y-b) : a, b \in k\} \cup \{(f) : f(x, y) \text{ irreducible}\} \cup \{0\}$, and these correspond to $\{(a, b) \in k \times k\} \cup \{\text{irred curves} \subset k \times k\} \cup \{k \times k\}$. The points (a, b) are the closed points, the others are called generic points for curves or the plane, because their closure is either everything (generic point of the plane) or are all of the points that lie on the curve.

Structure Sheaf: For $P \in \operatorname{Spec}(A)$, set $A_P = \{a/f : a, f \in A, f \notin P\}$. For $U = \operatorname{Spec}(A)$, open define, $\mathscr{O}(U) = \{s : U \to \coprod_{P \in U} A_P : s(P) \in A_P$ and s is locally a quotient$\}$, that is, $\forall P \in U$, there exists an open neighborhood V, $P \in V \subset U$ and $a, f \in A$ such that $s(Q) = a/f \in A_Q$ for all $Q \in V$.

Note: (1) $\mathscr{O}$ is a shead of rings on $\operatorname{Spec}(A)$

(2) The $\mathscr{O}(U)$ are not "really" functions, so care must be taken.

Definition 6.2 (Spectrum of a Ring)**.** *The spectrum of A is* $(\operatorname{Spec}(A), \mathscr{O})$.

Definition 6.3. *For* $f \in A$*, set* $D(f) = \{P \in \operatorname{Spec}(A) : f \notin P\} = \operatorname{Spec}(A) \setminus V(f)$.

Note: $\{D(f)\}$ is a basis for the topology on $\operatorname{Spec}(A)$. Let $U \subseteq \operatorname{Spec}(A)$ be open, $P \in U$. Write $\operatorname{Spec}(A) \setminus U = V(I)$. Then $P \notin V(I)$, so $I \not\subset P$, thus $\exists f \in I, f \notin P$, so $P \in D(f) \subset U$

Proposition 6.2. *1.* $\mathscr{O}_P = A_P$ *for all* $P \in \operatorname{Spec}(A)$

2. $\mathscr{O}(D(f)) = A_f$*, for all* $f \in A$*.*

3. $\mathscr{O}(\operatorname{Spec}(A)) = A$*.*

Proof. 1. For $P \in U$ open, we have a ring homomorphism $\mathscr{O} \to A_P$ by $s \mapsto s(P)$. This induces $\varphi : \mathscr{O}_P \to A_P$ by $\varphi(s_P) = s(P)$. This is surjective, as we can let $a/f \in A_P$. Then $a/f \in \mathscr{O}(D(f))$ and $\varphi(a/f) = a/f$. It is injective, as we can assume that $\varphi(s_P) = 0 \in A_P$, where $s \in \mathscr{O}(U)$, U

open, $P \in U$. WLOG, $s = a/f$ for all $Q \in U$, $a/f = 0 \in A_P$, so $f \notin P$ and there exists $g \in A \setminus P$ such that $ga = 0 \in A$. $s|_{U \cap D(g)} = \frac{ag}{fg} = 0 \in \mathscr{O}(U \cap D(g)) \Rightarrow s_P = 0 \in \mathscr{O}_P$.

2. Define $\psi : A_f \to \mathscr{O}(D(f))$ by $\psi(a/f^n) = [P \mapsto a/f_n \in A_P]$. For injectivity, assume that $\psi(a/f^n) = 0 \in \mathscr{O}(D(f))$. Set $I = \text{Ann}(a) \subset A$, for $P \in D(f)$, we have $a/f^n = 0 \in A_P$, so there exists $H \in A \setminus P$ such that $ha = 0$. Thus $h \in I$, $h \notin P$, so $P \notin V(I)$. Therefore, $D(f) \cap V(I) = \emptyset$, so $V(I) \subset V(f)$, thus $f \in \sqrt{I}$, so $f^m \in \text{Ann}(a)$, so $a/f^n = 0 \in A_f$.

   For surjectivity, let $s \in \mathscr{O}(D(f))$. There exists an open cover $D(f) = \cup V_i$ such that $s = a_i/g_i$ on V_i for all i. WLOG, $V_i = D(h_i)$ for $h_i \in A$. $D(h_i) \subset D(g_i)$ implies that $V(h_i) \supseteq V(g_i)$, so $h_i \in \sqrt{(g_i)}$. Thus $h_i^{n_i} = c_i g_i$ with $c_i \in A$. Note that $a_i/g_i = c_i a_i / c_i g_i = c_i a_i / h_i^{n_i}$. So we replace a_i with $c_i a_i$ and h_i with $h_i^{n_i}$, and WLOG, $s = a_i/h_i$ pm $D(h_i)$ for all i. We claim that $D(f)$ =union of finitely many $D(h_i)$. $D(f) \subseteq \cup D(h_i)$ iff $V(f) \supseteq \cap V(h_i) = V(\langle h_i \rangle)$ iff $f \in \sqrt{\langle h_i \rangle}$ iff $f^m \in \langle h_i \rangle$ iff $f^m = \sum b_i h_i$ for a finite sum with $b_i \in A$. Note that $s = a_i/h_i = a_j/h_j$ on $D(h_i) \cap D(h_j) = D(h_i h_j)$. ψ injective implies $a_i/h_i = a_j/h_j \in A_{h_i h_j}$, so $(h_i h_j)^n (h_j a_i - h_i a_j) = 0$ That is, $h_j^{n+1}(h_i^n a_i) = h_i^{n+1}(h_j^n a_j)$ for all i, j. Replace a_i with $h_i^n a_i$ and h_i with h_i^{n+1}. WLOG, $h_j a_i = h_i a_j \in A$.

   $f^m = \sum b_i h_i$, set $a = \sum b_i a_i$, $h_j a = \sum_i b_i a_i h_j = \sum b_i a_j h_i = f^m a_j$, so $s = a_j/h_j = a/f^m$ on $D(h_i)$, the $D(h_i)$ cover $D(f)$, so $s = a/f^m \in \mathscr{O}(D(f))$.

3. Follows from part 2.

□

Definition 6.4 (Ringed Space). *A Ringed Space is a pair $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings on X. (Usually just denoted by X).*

A morphism of ringed spaces $f : X \to Y$ is a pair $f = (f, f^\sharp)$ where $f : X \to Y$ is continuous and $f^\sharp : \mathscr{O}_Y \to f_* \mathscr{O}_X$ a ring homomorphism.

Let $f : X \to Y$ be a morphism of ringed spaces. $P \in X$. For $V \subset Y$ open, $f(P) \in V$, then we have $\mathscr{O}_Y(V) \xrightarrow{f^\sharp} f_* \mathscr{O}_X(V) = \mathscr{O}_X(f^{-1}(V)) \to \mathscr{O}_{X,P}$ induces $f_P^\sharp : \mathscr{O}_{Y,f(P)} \to \mathscr{O}_{X,P}$.

Definition 6.5 (Locally Ringed Space). *$X = (X, \mathscr{O}_X)$ is a locally ringed space if it is a ringed space such that $\mathscr{O}_{X,P}$ is a local ring for all $P \in X$. We call the maximal ideal $\mathfrak{m}_P \subseteq \mathscr{O}_{X,P}$.*

$f : X \to Y$ is a morphism of locally ringed spaces iff it is a morphism of ringed spaces such that $f_P^\sharp : \mathscr{O}_{Y,f(P)} \to \mathscr{O}_{X,P}$ is a local homomorphism for all $P \in X$, that is, $f_P^\sharp(\mathfrak{m}_{f(P)}) \subseteq \mathfrak{m}_P$. (iff $(f_P^\sharp)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{f(P)}$).

Example: $(\text{Spec}\, A, \mathscr{O})$ is a locally ringed space.

Why do we want locally ringed spaces? Look at all the morphisms $\text{Spec}(A) \to \text{Spec}(B)$, there are a lot of morphisms of ringed spaces, however, the morphisms of locally ringed spaces are in correspondence with ring homomorphisms $B \to A$.

Set $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$. Let $f : A \to B$ be a ring homomorphism. Define $\varphi : Y \to X$ by $\varphi(Q) = f^{-1}(Q)$.

Continuous: $I \subset A$ an ideal, $\varphi^{-1}(V(I)) = \{Q \in Y : f^{-1}(Q) \supset I\} = V(f(I)B) \subset Y$

For $Q \in Y$, let $f_Q : A_{f^{-1}(Q)} \to B_Q$ be a map on local rings. Let $U \subset X$ open, $\mathscr{O}_X(U) = \{s : U \to \coprod_{p \in U} A_p | s$ is locally a quotient$\}$. Define $\varphi^\sharp : \mathscr{O}_X(U) \to \varphi_* \mathscr{O}_Y(U) = \mathscr{O}_Y(\varphi^{-1}(U))$ by $s \mapsto [Q \mapsto f_Q(s(\varphi(Q))) \in B_Q]$

Check: $\varphi^\sharp_Q = f_Q : \mathscr{O}_{X,\varphi(Q)} \to \mathscr{O}_{Y,Q}$.

Proposition 6.3. *There is a bijective correspondence between ring homomorphisms $f : A \to B$ and morphisms of LRS $\varphi : Y \to X$ by $f \mapsto \varphi$.*

Proof. Note: $f = \varphi^\sharp : \Gamma(X, \mathscr{O}_X) = A \to \Gamma(Y, \mathscr{O}_Y) = B$.

Must Show: Any morphism of LRS $\varphi : Y \to X$ is determined by $\varphi^\sharp : A \to B$.

Let $Q \in Y$, set $P = \varphi(Q) \in X$. The following commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi^\sharp} & B \\ \downarrow & & \downarrow \\ A_P = \mathscr{O}_{X,P} & \xrightarrow{\varphi^\sharp_Q} & \mathscr{O}_{Y,Q} = B_Q \end{array}$$

$\varphi^\sharp_Q$ is a local ring homomorphism so $\mathfrak{m}_P = (\varphi^\sharp_Q)^{-1}(\mathfrak{m}_Q)$, so $P = (\varphi^\sharp)^{-1}(Q)$ $\square$

Remark: If X is an LRS, $U \subseteq X$ open, then $(U, \mathscr{O}_X|_U)$ is an LRS and $U \to X$ the inclusion is a morphism of LRS.

Definition 6.6 (Affine Scheme)**.** *An affine scheme is an LRS $(X, \mathscr{O}_X)$ such that $(X, \mathscr{O}_X) \simeq \operatorname{Spec}(A)$ for some ring A.*

Definition 6.7 (Scheme)**.** *A scheme is an LRS $(X, \mathscr{O}_X)$ such that there is an open cover $X = \cup U_\alpha$ such that $(U_\alpha, \mathscr{O}_X|_{U_\alpha})$ is an affine scheme for all α.*

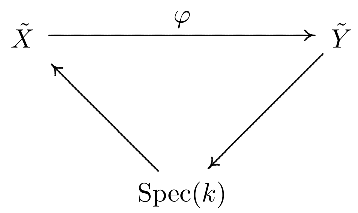
Example: If k is a field, then $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$, $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$, and the dimension of $\mathbb{A}^n_{\mathbb{Z}}$ is $n + 1$.

Example: Assume that $\operatorname{char}(k) = p > 0$, and A any k-algebra. Then there is a ring homomorphism $A \to A : a \mapsto a^p$. This gives the Absolute Frobenius Morphism $F : \operatorname{Spec}(A) \to \operatorname{Spec}(A)$, $F(P) = \{a \in A : a^p \in P\} = P$, so it is the identity on points, but is NOT the identity of schemes.

Example: Suppose $k = \bar{k}$, X a (pre)variety over k. Define $\tilde{X} = \{$closed irreducible subvarieties in $X\}$. Then $i : X \to \tilde{X}$ is the inclusion. Open subsets of $\tilde{X}$ are $\tilde{U}$ for $U \subseteq X$ open. Define $\mathscr{O}_{\tilde{X}} = i_* \mathscr{O}_X$.

Exercise: $(\tilde{X}, \mathscr{O}_{\tilde{X}})$ is a scheme.

Note: There exists a unique morphism of varieties $X \to \{pt\} = \operatorname{Spec}(k)$, and this gives a structure morphism from $\tilde{X} \to \operatorname{Spec}(k)$. ($\tilde{X}$ is a scheme over $\operatorname{Spec}(k)$)



If $\varphi : X \to Y$ is a morphism of varieties, then commutes.

Exercise: Morphisms of varieties $X \to Y$ correspond to morphisms of schemes as above.

Example: The absolute Frobenius map $F : \tilde{X} \to \tilde{X}$ is NOT a morphism of varieties.

Definition 6.8 (Reduced Scheme)**.** *A scheme X is reduced if $\mathscr{O}_X(U)$ is a reduced ring for all $U \subset X$ open.*

Definition 6.9 (Finite Type)**.** *A morphism $\varphi : X \to Y$ of schemes is of finite type if for all open affine $V \subset Y$ there exists a finite open affine cover $\varphi^{-1}(V) = \cup U_\alpha$ such that $\mathscr{O}_X(U_\alpha)$ is a finitely generated $\mathscr{O}_Y(V)$-algebra for all α.*

Exercise: The category of prevarieties over an algebraically closed field k is equivalent to the reduced schemes of finite type over $\mathrm{Spec}(k)$.

<u>Projective Schemes</u>

$S = \oplus_{d\geq 0} S_d$ graded ring, $S_d \cdot S_e \subseteq S_{d+e}$. Exercise: $1 \in S_0$.

$I \subseteq S$ a homogeneous ideal, if $I = \oplus(I \cap S_d)$ iff I is generated by homogeneous elements. EG $S_+ = \oplus_{d>0} S_d$.

Definition 6.10. $\mathrm{Proj}(S) = \{P \subseteq S$ *homogeneous prime such that* $S_+ \not\subset P\}$

For I a homogeneous ideal, set $V(I) = \{P \in \mathrm{Proj}(S) | P \supseteq I\}$.

Check $V(I) \cup V(J) = V(IJ) = V(I \cap J)$. $\cap V(I_\alpha) = V(\sum I_\alpha)$

Topology: $U \subset \mathrm{Proj}(S)$ open iff $\mathrm{Proj}(S) \setminus U = V(I)$.

Let $P \in \mathrm{Proj}(S)$, set $T = \{$homogeneous elements of $S \setminus P\}$, $T^{-1}S = \{a/f : a \in S, f \in T\} = \oplus_{d\in\mathbb{Z}}(T^{-1}S)_d$, $\deg(a/f) = \deg a - \deg f$. Define $S_{(P)} = (T^{-1}S)_0$

For $U \subset \mathrm{Proj}(S)$ open, define $\mathscr{O}(U) = \{s : U \to \coprod_{p\in U} S_{(p)} | s(p) \in S_{(p)}$ and is locally a quotient $\}$.

ie, for all $P \in U$, there is an open nbhd $P \in V \subset U$ and homogeneous elements $a, f \in S$ of the same degree such that $s(Q) = a/f \in S_{(Q)}$ for all $Q \in V$.

Note: $S_{(P)}$ is a local ring, $S_{(P)} \subset T^{-1}S \to S_P$, max ideal $S_{(P)} \cap PS_P$.

Proposition 6.4. *S graded ring,* $(\mathrm{Proj}(S), \mathscr{O})$ *as above,* $\mathscr{O}_P = S_{(P)}$ *for all* $P \in \mathrm{Proj}(S)$.

Proof. $\mathscr{O}_P \to S_{(P)}$, $t_P \mapsto t(P)$, and this is injective and surjective, following the proof for $\mathrm{Spec}\, A$. □

For $f \in S_+$ homogeneous, define $D_+(f) = \{P \in \mathrm{Proj}(S) : f \notin P\} = \mathrm{Proj}(S) \setminus V(f)$

Exercise: $\{D_+(f)\}$ is a basis for the topology on $\mathrm{Proj}(S)$.

Define: $S_{(f)} = (S_f)_0 = \{a/f^n : \deg(a) = n \deg(f)\}$.

Proposition 6.5. $(D_+(f), \mathscr{O}|_{D_+(f)}) \simeq \mathrm{Spec}(S_{(f)})$.

Proof. $S \to S_f \supseteq S(f)$. Define $\varphi : D_+(f) \to \mathrm{Spec}(S_{(f)})$ by $P \mapsto PS_f \cap S_{(f)}$.

Exercise: For $Q \in \mathrm{Spec}(S_{(f)})$ we have $\sqrt{\langle Q \rangle} \subset S_f$ prime ideal.

Inverse map $\varphi^{-1}(Q) = \sqrt{\langle Q \rangle} \cap S$.

Homeomorphism: $D_+(h) \subseteq D_+(f) \xrightarrow{\varphi} \mathrm{Spec}(S_{(f)})$ sends $D_+(h)$ to $D(h^{\deg(f)}/f^{\deg(h)})$.

Note: $(S_{(f)})_{\varphi(P)} \simeq S_{(P)}$.

For $V \subset \mathrm{Spec}(S_{(f)})$ open, then $\varphi_* \mathscr{O}_{\mathrm{Proj}}(V) = \mathscr{O}_{\mathrm{Proj}}(\varphi^{-1}(V)) = \{s : \varphi^{-1}(V) \to \coprod_{P \in \varphi^{-1}(V)} S_{(P)}\} = \{s : V \to \coprod_{Q \in V} (S_{(f)})_Q\} = \mathscr{O}_{\mathrm{Spec}\, S_{(f)}}(V)$. □

Example: A a ring, $\mathbb{P}^n_A = \mathrm{Proj}\, A[x_0, \ldots, x_n]$, covered by

$$D_+(x_i) = \mathrm{Spec}\, A[x_0, \ldots, x_n]_{(x_i)} = \mathrm{Spec}\, A[x_0/x_i, \ldots, x_n/x_i] = \mathbb{A}^n_A$$

Definition 6.11 (Properties of Schemes)**.** *A scheme X is*

1. *connected if it is connected as a topological space*
2. *irreducible if it is irreducible as a topological space*
3. *reduced if $\mathscr{O}_X(U)$ is a reduced ring for all open $U \subset X$ iff $\mathscr{O}_{X,P}$ reduced for all $P \in X$.*
4. *integral if $\mathscr{O}_X(U)$ a domain for all $U \subset X$ open*
5. *noetherian if $X = \cup \mathrm{Spec}(A_i)$ where A_i Noetherian and the cover is finite*
6. *locally noetherian if $X = \mathrm{Spec}(A_i)$ for A_i noetherian, not necessarily a finite cover*

Example: $X = \mathrm{Spec}(A)$ is irreducible iff $\sqrt{0} \subset A$ is prime, it is reduced iff $\sqrt{0} = 0$ and it is integral iff A is a domain.

Proposition 6.6. *A scheme X is integral iff X is irreducible and reduced.*

Note: An open subscheme of (locally) Noetherian scheme is (locally) Noetherian

Proposition 6.7. $\mathrm{Spec}(A)$ *is locally noetherian iff notherian iff A is noetherian*

Proof. Assume that $\mathrm{Spec}(A)$ is locally noetherian. Let $U = \mathrm{Spec}(B) \subset \mathrm{Spec}(A)$ be open, B is noetherian. There exists $f \in A$ such that $D(f) \subseteq U$, $f|_U \in \mathscr{O}(U) = B$. $\mathrm{Spec}(A_f) = \mathrm{Spec}(B_{f|_U}) = D(f)$. $A_f = B_{f|_U}$ Noetherian.

We can write $\mathrm{Spec}(A) = \cup_i \mathrm{Spec}(A_{f_i})$, A_{f_i} Noetherian, so $\emptyset = V(\{f_i\}) \Rightarrow (f_i) = (1) = A \Rightarrow 1 = \sum a_i f_i$ with $a_i \in A$ a finite sum.

Thus, there are $f_1, \ldots, f_r \in A$ such that A_{f_i} noetherian and $(f_1, \ldots, f_r) = A$.

Let $I \subseteq A$ ideal, choose $g_1, \ldots, g_m \in I$ such that $I_{f_i} = IA_{f_i} = (g_1, \ldots, g_m) \subseteq A_{f_i}$ for all i. Claim: $I = (g_1, \ldots, g_m)$. $b \in I$, $f_i^N b \in (g_1, \ldots, g_m)$ for all i. So $(f_1^N, \ldots, f_r^N) = (1) \subseteq A$, so A is Noetherian. □

Definition 6.12 (Properties of Morphisms)**.** *A morphism $f : X \to Y$ is*

1. *locally of finite type if for all open affine $V \subseteq Y$ there exists open affine cover $f^{-1}(V) = \cup U_i$ s.t. $\mathscr{O}_X(U_i)$ is finitely generated $\mathscr{O}_Y(V)$-algebra.*
2. *of finite type if $\exists$ a finite cover of $f^{-1}(V)$ as above.*
3. *affine if for all open affine $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open affine.*
4. *is finite if affine and $\mathscr{O}_X(f^{-1}(V))$ is a finitely generated $\mathscr{O}_Y(V)$-module whenever V is affine.*

Exercise: In all cases, it is enough to know the property on a single open affine cover.

Definition 6.13 (Open Immersion)**.** *$f : X \to Y$ is an open immersion if it can be factored $f : X \xrightarrow{\simeq} U \subseteq Y$ open.*

Definition 6.14 (Closed Immersion/Closed Subscheme)**.** *$f : X \to Y$ is a closed immersion if*

1. *f is a homeomorphism of X with closed subset of Y.*
2. *$f^\sharp : \mathscr{O}_Y \to f_* \mathscr{O}_*$ is surjective.*

Example: $Y = \operatorname{Spec}(A)$, $I \subseteq A$ ideal, $A \to A/I$ gives a closed immersion $\operatorname{Spec}(A/I) \to \operatorname{Spec}(A)$ with image $V(I)$.

Exercise: All closed immersions $X \to \operatorname{Spec}(A)$ have this form.

Exercise: Y scheme, $V \subseteq Y$ a closed subset, then there exists a unique closed immersion $X \to Y$ with image V such that X is reduced.

Definition 6.15 (Dimension)**.** $\dim(X) =$*supremum of n such that $\exists\, \emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n \subseteq X$ with Z_i closed irreducible subset.*

$Z \subseteq X$ closed irreducible. Then $\operatorname{codim}(Z; X)$ *is the supremum of n such that $\exists Z = Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n \subseteq X$ with Z_i closed irreducible.*

$Y \subseteq X$ is any closed subset, then $\operatorname{codim}(Y; X) = \inf\{\operatorname{codim}(Z; X) : Z \subseteq Y$ *closed irreducible*$\}$.

WARNING: $Z \subseteq X$ closed and irreducible, $\dim(Z) + \operatorname{codim}(Z; X) = \dim(X)$ does not always hold!

Products

Definition 6.16 (Product)**.** *Let X, Y, S be schemes with morphisms $\alpha : X \to S$ and $\beta : Y \to S$. A product of X and Y over S is a scheme $X \times_S Y$ with morphisms*

$$\begin{array}{ccc} & X \times_S Y & \\ \swarrow p & & \searrow q \\ X & & Y \\ \searrow \alpha & & \swarrow \beta \\ & S & \end{array}$$

along with the universal property that for any scheme Z with morphisms f, g to X and Y s.t. $\alpha f = \beta g$, there exists a unique morphism $\varphi : Z \to X \times_S Y$ such that $f = p\varphi$ and $g = q\varphi$.

Exercise: $X \times_S Y$ is unique up to unique isomorphism.

Exercise: X any scheme, A is a commutative ring, then there is a correspondence between $\{\text{morphisms } X \to \operatorname{Spec}(A)\}$ and $\{\text{ring hom } A \to \mathscr{O}_X(X)\}$.

Consequence: $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(R)$, then α, β make A and B into R-algebras, and $X \times_S Y = \operatorname{Spec}(A \otimes_R B)$.

Observe that if $S \subseteq T$ is an open subscheme, then $X \times_T Y = X \times_S Y$, as if $j : S \to T$ is the inclusion, then $\alpha f = \beta g \iff j\alpha f = j\beta g$. Also observe that if $U \subseteq X$ open, then $U \times_S Y = p^{-1}(U) \subseteq X \times_S Y$, and so it is an open subscheme of $X \times_S Y$, because if $h : Z \to X$ is a morphism, so that $h(Z) \subseteq U$, then we can factor $h : Z \to U \subseteq X$.

Consequence: If $X' \subseteq X$ and $Y' \subseteq Y$, $S' \subseteq S$ are all open such that $\alpha(X'), \beta(Y') \subseteq S'$, then $X' \times_{S'} Y' = X' \times_S Y'$ and this is $p^{-1}(X') \cap q^{-1}(Y') \subseteq X \times_S Y$.

Construction

Assume that S is affine. Take open affine covers $X = \cup X_i$, $Y = \cup Y_j$. We glue $X \times_S Y := \bigcup_{i,j} X_i \times_S Y_j$ by $(X_i \times_S Y_j) \cap (X_k \times_S Y_\ell) = (X_i \cap X_k) \times_S (Y_j \cap Y_\ell)$.

If S is any scheme, we take an open affine cover $S = \cup S_i$. $X_i = \alpha^{-1}(S_i)$, $Y_i = \beta^{-1}(S_i)$ in X and Y are open sets. And here we glue $X \times_S Y := \cup_i X_i \times_{S_i} Y_i$ by $(X_i \times_{S_i} Y_i) \cap (X_j \times_{S_j} Y_j) = (X_i \cap X_j) \times_{S_i \cap S_j} (Y_i \cap Y_j)$

Examples: X is a scheme

1. $U, V \subseteq X$ open. $U \times_X V = U \cap V$.
2. $Y, Z \subseteq X$ closed subschemes. We define $Y \cap Z := Y \times_X Z$, the scheme theoretic intersection. This is still a closed subscheme of X.

$X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, $Y = V(y - x^2) = \operatorname{Spec} k[x, y]/(y - x^2)$, $Z = V(y) = \operatorname{Spec} k[x, y]/(y)$. We have a diagram of commutative rings

$$\begin{array}{ccc} & k[x,y] & \\ \swarrow & & \searrow \\ k[x,y]/(y - x^2) & & k[x,y]/(y) \\ \searrow & & \swarrow \\ & k[x,y]/(y - x^2, y) & \end{array}$$

So $Y \times_X Z = Y \cap Z = \operatorname{Spec} k[x,y]/(x^2, y) = \operatorname{Spec} k[x]/(x^2)$. $\dim_k k[x]/(x^2) = 2$.

Let X be a scheme and $p \in X$ a point.

Definition 6.17 (Residue Field at p)**.** *$k(P) = \mathscr{O}_{X,P}/\mathfrak{m}_P$ is called the residue field at P.*

If $P \in U = \operatorname{Spec}(A) \subseteq X$ open, then $k(P) = A_P/PA_P$, the field of fractions of A/P.

Note: $A \to k(P)$ gives a morphism $\operatorname{Spec}(k(P)) \to \operatorname{Spec}(A) \to X$ with image $\{P\}$.

Examples: X an irreducible algebraic variety over k. If $P \in X$ is a closed point, we get $k(P) = k$. If $P_0 \in X$ is a generic point, we get $k(P_0) = k(X)$.

For $p \in U \subseteq X$, $f \in \mathscr{O}_X(U)$, set $f(P)$ =the image of f under $\mathscr{O}_X(U) \to \mathscr{O}_{X,P} \to k(P)$.

Note: $V(f) = \{P \in U : f(P) = 0 \in k(P)\}$ is relatively closed in U.

Let $\varphi : Y \to X$ a morphism, $\varphi(Q) = P$, $P \in U$. Then $\varphi_Q^\sharp : \mathscr{O}_{X,P} \to \mathscr{O}_{Y,Q}$ is a local ring homomrophism so it induces $\bar{\varphi}_Q^\sharp : k(P) \to k(Q)$ a field extension. $\varphi^\sharp(f) \in \mathscr{O}_Y(\varphi^{-1}(U))$, $\varphi^\sharp(f)(Q) = \bar{\varphi}_Q^\sharp(f(P)) = \bar{\varphi}_Q^\sharp(f(\varphi(Q))) \in k(Q)$.

E.G. X, Y varieties over $k = \bar{k}$, $Q \in Y$ a closed point implies that $\varphi^\sharp(f)(Q) = f(\varphi(Q)) \in k$.

Exercise: A rational map of irreducible varieties $f : X \dashrightarrow Y$ is the same as a morphism of schemes over k $f : \operatorname{Spec} k(X) \to Y$

$P \in X$ a closed point, then $\{$rational maps $X \dashrightarrow Y$ defined at $P\}$ correspond to $\{$morphismsSpec $\mathscr{O}_{X,P} \to Y$ over $k\}$.

Note: $\operatorname{Spec}(k(X)) \to \operatorname{Spec}(\mathscr{O}_{X,P}) \to X$.

Examples of Products

X, Y varieties over k, the product $X \times Y$ from last semester corresponds to $X \times_k Y = X \times_{\operatorname{Spec}(k)} Y$.

Fibers: $\varphi : X \to Y$ morphism of schemes, $Q \in Y$, then $X_Q = \varphi^{-1}(Q) := X \times_Y \operatorname{Spec} k(Q)$ with the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \uparrow & & \uparrow \\ X_Q & \longrightarrow & \operatorname{Spec}(k(Q)) \end{array}$$

Example: $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$, $\varphi(t) = t^2$. For $a \in \mathbb{A}^1$ a closed point, $\varphi^{-1}(a) = \mathbb{A}^1 \times_{\mathbb{A}^1} \{a\} = \operatorname{Spec}(k[t] \otimes_{k[t^2]} k[t^2]/(t^2 - a)) = \operatorname{Spec}(k[t]/(t^2 - a))$, if $a \neq 0$, we get $\varphi^{-1}(a) = \{\sqrt{a}, -\sqrt{a}\}$, if $a = 0$, $\varphi^{-1}(0) = \operatorname{Spec} k[t]/(t^2)$.

Example: $\mathbb{P}_k^2 \times_k \mathbb{A}_k^1 = \operatorname{Proj} k[x,y,z] \times \operatorname{Spec} k[t] = \operatorname{Proj} A[x,y,z]$ where $A = k[t]$.

$E = V(zy^2 - x(x-z)(x-tz)) \subseteq \mathbb{P}^2 \times \mathbb{A}^1$, $\varphi : E \to \mathbb{A}^1$ projection, then for $\lambda \in \mathbb{A}^1$ a closed point, $E_\lambda = V(zy^2 - x(x-z)(x-\lambda z)) \subseteq \mathbb{P}_k^2$.

Separated Morphisms

Let $f : X \to Y$ a morphism, $\Delta : X \to X \times_Y X$ the diagonal morphism (the unique map into this product which, when composed with either projection, is the identity)

Definition 6.18 (Separated)**.** *f is separated if $\Delta : X \to X \times_Y X$ is a closed immersion.*

X is separated if $X \to \operatorname{Spec}(\mathbb{Z})$ is separated.

Note: A a ring, then there exists a unique $\mathbb{Z} \to A$.

Example: $X = (\mathbb{A}^1 \setminus \{0\}) \cup \{0_1, 0_2\}$ the doubled line to $\operatorname{Spec}(k)$ is not separated (See last semester).

Proposition 6.8. *Any morphism $f : X \to Y$ of affine schemes is separated.*

Proof. $X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B), f^\sharp : B \to A$. Then $X \times_Y X = \operatorname{Spec}(A \otimes_B A)$. $\Delta : X \to X \times_Y X$ corresponds to $\Delta^\sharp : A \otimes_B A \to A$, $a_1 \otimes a_2 \mapsto a_1 a_2$.

$\Delta^\sharp$ is surjective, so $A = A \otimes_B A/I$, so $X = V(I) \subset X \times_Y X$. □

Corollary 6.9. *$f : X \to Y$ separated iff $\Delta(X) \subseteq X \times_Y X$ closed.*

Proof. Assume that $\Delta(X)$ is closed. It is a homeomorphism as $\Delta : X \to \Delta(X)$ has continuous inverse $\Delta(X) \to X \times X \to X$ by p_1.

We must now check that $\Delta^\sharp : \mathscr{O}_{X \times_Y X} \to \Delta_* \mathscr{O}_X$ is surjective.

If $Q \in X \times_Y X \setminus \Delta(X)$, then $\mathscr{O}_{X \times_Y X, Q} \to (\Delta_* \mathscr{O}_X)_Q = 0$ is surjective. Let $P \in X$. Choose $V \subseteq Y$ open affine such that $f(P) \in V$. Choose $U \subseteq f^{-1}(U)$ open affine such that $P \in U$. Then $\Delta(P) \in U \times_V U \subseteq X \times_Y X$ so $\Delta^\sharp$ is surjective in a nbhd of $\Delta(P)$, because $\Delta : U \to U \times_V U$ is separated. □

Let $(G, \leq)$ be a totally ordered abelian group. IE, $g_1 \leq g_2$ implies $g_1 + g_3 \leq g_2 + g_3$. K a field, and $K^\times = K \setminus \{0\}$.

Definition 6.19 (Valuation)**.** *A valuation of K with values in G is a map $v : K^\times \to G$ s.t. $v(xy) = v(x) + v(y)$, $v(x + y) \geq \min\{v(x), v(y)\}$.*

e.g. $v : k(t) \to \mathbb{Z}$ by $v(t^m f(t)) = m$ if f is defined at 0 and $f(0) \neq 0$

Note: $\{x \in K^\times : v(x) \geq 0\} \cup \{0\} \subseteq K$ is a subring.

Definition 6.20 (Valuation Ring)**.** *R is a valuation ring if R is a domain and there exist a valuation $v : K(R)^\times \to G$ for some G such that $R = \{x \in K(R)^\times : v(x) \geq 0\} \cup \{0\}$.*

$R \subseteq K(R)$ gives us a morphism $i : \operatorname{Spec} K(R) \to \operatorname{Spec} R$.

Theorem 6.10. *$f : X \to Y$ is a morphism, then f is separated iff the following condition holds: For any valuation ring R and morphisms $\alpha : \operatorname{Spec} R \to Y$ and $\beta : \operatorname{Spec} K(R) \to X$ such that $\alpha i = f \beta$*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \beta \uparrow & & \uparrow \alpha \\ \operatorname{Spec} K(R) & \xrightarrow{i} & \operatorname{Spec}(R) \end{array}$$

Then there is at most one $\gamma : \operatorname{Spec} R \to X$ making this all commute, that is, $f\gamma = \alpha$ and $\gamma i = \beta$.

Intuition: X, Y are prevarieties, C a curve and $P \in C$ a nonsingular point. $R = \mathscr{O}_{C,P}$ is a DVR. $K = k(C)$. Then $\beta : C \dashrightarrow X$ is a rational map, and $\alpha : C \dashrightarrow Y$ is also a rational map defined at $P \in C$. Then $\alpha = f\beta$ is the commutativity condition on the square. If f is separated, then there is at most one possible value of $\beta(P)$.

e.g. If X is the doubled affine line, Y is $\operatorname{Spec} k$ then $\beta : \mathbb{A}^1 \dashrightarrow X$ defined on $\mathbb{A}^1 \setminus \{0\}$. There are two ways to define $\beta(0)$, so this is not separated.

Corollary 6.11. *1. open and closed immersions are separated*

2. compositions of separated morphisms are separated.

3. separated morphisms are stable under base extensions

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ X' = X \times_Y Y' & \xrightarrow{f'} & Y' \end{array}$$

Definition 6.21 (Base Extension)**.** *Then f' is the base extension of f by Y'.*

Proof. We will prove (b).

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ are separated morphisms. Let R be a valuation ring, take $\beta : \operatorname{Spec} K(R) \to X$ and $\alpha : \operatorname{Spec}(R) \to Z$.

Assume that $\gamma_1, \gamma_2 : \operatorname{Spec} R \to X$ are morphisms satisfying the diagram. Proof by diagram chasing. □

Exercise: If g is separated, then gf is separated iff f is separated.

Let X be a scheme, $x_1 \in X$.

Definition 6.22 (Specialization)**.** *$x_0 \in X$ is a specialization of x_1 if $x_0 \in \overline{\{x_1\}}$.*

This gives a ring homomorphism $\mathscr{O}_{X,x_0} \to \mathscr{O}_{X,x_1}$ which is NOT local.

Note: $\operatorname{Spec} k(x_0) \to X$ has image $\{x_0\}$, $\operatorname{Spec} k(x_0) \to \operatorname{Spec} \mathscr{O}_{X,x_0} \to X$ has image $\{x_1 \in X : x_0 \text{ is a specialization of } x_1\}$.

Assume R is a local domain, $K = K(R)$ $i : \operatorname{Spec} K \to \operatorname{Spec} R$, $t_0 = \mathfrak{m}_R \in \operatorname{Spec} R$ and $t_1 = 0 \in \operatorname{Spec} R$. Let $\beta : \operatorname{Spec} K \to X$ a morphism, $x_1 = \beta(0)$, $\beta^\sharp : k(x_1) \to K$

Lemma 6.12. *There is a correspondence $\{\gamma : \operatorname{Spec} R \to X | \gamma i = \beta\}$ to $\{x_0 \in \overline{\{x_1\}} | \text{image } \mathscr{O}_{X,x_0} \to \mathscr{O}_{X,x_1} \to k(x_1) \xrightarrow{\beta^\sharp} K \text{ is a subset of } R\}$ by $\gamma \mapsto \gamma(t_0)$.*

Proof. Assume $\gamma i = \beta$. $t_0 \in \overline{\{t_1\}}$ implies that $\gamma(t_0) \in \overline{\{x_1\}}$. $\mathfrak{m}_{x_0} \mapsto \mathfrak{m}_R$. We get the following commutative square:

$$\begin{array}{ccc} \mathscr{O}_{X,x_0} & \xrightarrow{\gamma^\sharp_{t_0}} & \mathscr{O}_{t_0} = R \\ \downarrow & & \downarrow \\ \mathscr{O}_{X,x_1} & \xrightarrow{\gamma^\sharp_{t_1}} & \mathscr{O}_{t_1} = K \end{array}$$

Assume that $x_0 \in \overline{\{x_1\}}$. Then define $\gamma : \operatorname{Spec} R \to \operatorname{Spec} \mathscr{O}_{X,x_0} \to X$. □

Recall: A domain R is a valuation ring of K if $K = K(R)$ and $\exists$ valuation $v : K^\times \to G$ such that $R = \{a \in K | v(a) \geq 0\} \cup \{0\}$.

Note: R is a local ring, $\mathfrak{m}_R = \{a \in K | v(a) > 0\}$.

Fact: Every local ring $R' \subseteq K$ is dominated by some valuation ring R of K (that is, $R' \subseteq R$ with $\mathfrak{m}_{R'} \subseteq \mathfrak{m}_R$). $R' \subseteq K$ a local subring is a valuation ring of K iff R' is not domainated by strictly larger $R \subseteq K$.

Theorem 6.13. *X Nötherian. $f : X \to Y$ a morphism, $\Delta : X \to X \times_Y X = P$. Then $\Delta(X) \subseteq P$ is closed iff for all valuation rings R with morphisms α, β with $\alpha i = f\beta$ there exists at most one γ sch that $f\gamma = \alpha$ and $\gamma i = \beta$.*

Proof. $\Rightarrow$: Let $\gamma_1, \gamma_2 : \operatorname{Spec} R \to X$ be given, with $\gamma_j i = \beta$, $f\gamma_j = \alpha$. Define $\varphi : \operatorname{Spec} R \to P$ by $\pi_j \varphi = \gamma_j$. Then $\gamma_j i = \beta$ imply that $\varphi(t_1) = \varphi(i(0)) \in \Delta(X)$ and $\varphi(t_0) \in \overline{\varphi(t_1)} \subseteq \overline{\Delta(X)} = \Delta(X)$. Thus, $\pi_1(\varphi(t_0)) = \gamma_1(t_0) = \gamma_2(t_0) = \pi_2(\varphi(t_0))$, so $\gamma_1 = \gamma_2$ by the lemma.

$\Leftarrow$: Let $z_1 \in \Delta(X)$ and $z_0 \in \overline{\{z_1\}} \subseteq P$. Claim: $z_0 \in \Delta(X)$.

Set $K = k(z_1)$, $R' = \operatorname{Im}(\mathscr{O}_{P,z_0} \to \mathscr{O}_{P,z_1} \to K) \subseteq K$. The fact implies that there is a valuation ring R of K such that $\mathscr{O}_{P,z_0} \to R \subseteq K$ is a local ring homomorphism. $\beta : \operatorname{Spec} K \to P$, $\pi_k : P \to X$.

Exercise: $z_1 \in \Delta(X)$ implies that $\pi_1\beta = \pi_2\beta : \operatorname{Spec} K \to X$.

Now the lemma implies that there is $\varphi : \operatorname{Spec} R \to P$ such that $\beta = \varphi i$, $\varphi(t_0) = z_0$. Set $\gamma_j = \pi_j\varphi : \operatorname{Spec} R \to X$. Then $f\gamma_1 = f\pi_1\varphi = f\pi_2\varphi = f\gamma_2$. So the algebraic assumption says that $\gamma_1 = \gamma_2$, so $\varphi = \Delta\pi_1\varphi$, thus $z_0 = \varphi(t_0) \in \Delta(X)$, so the claim holds.

So now X Nötherian implies that $X = X_1 \cup \ldots \cup X_n$ where X_i is closed and irreducible.

So $X_i = \overline{\{x_i\}}$, $x_i \in X$, we set $z_i = \Delta(x_i) \in P$. So $X \subseteq \overline{\{x_1, \ldots, x_n\}}$, thus $\Delta(X) \subseteq \overline{\{z_1, \ldots, z_n\}} = \overline{\{z_1\}} \cup \ldots \cup \overline{\{z_n\}} \subseteq \Delta(X)$ by the claim, and so the theorem holds. □

Definition 6.23 (Properness)**.** *$f : X \to Y$ is proper if f is separated, of finite type, and f is universally closed, which means that any base extension of f is a closed map. (takes closed sets to closed sets.)*

Definition 6.24 (Redefinition of a Variety)**.** *Let k be a field. A prevariety over k is a reduced scheme of finite type over k. A variety is a separated pre-variety X, that is, the structure morphism $X \to \operatorname{Spec}(k)$ is separated.*

Definition 6.25 (Complete)**.** *A variety is complete if $X \to \mathrm{Spec}(k)$ is proper. ie, $X \times_k Y \to Y$ is closed for all Y.*

Theorem 6.14. *X Nötherian, $f : X \to Y$ of finite type. Then f is proper iff $\forall$ valuation rings R with a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \beta \uparrow & \nwarrow \exists! \gamma & \uparrow \alpha \\ \mathrm{Spec}\, K(R) & \xrightarrow{i} & \mathrm{Spec}\, R \end{array}$$

there $\exists! \gamma$ such that $f\gamma = \alpha, \gamma i = \beta$

Corollary 6.15. *X a complete variety, C a curve, $P \in C$ a nonsingular point, then $f : C \setminus \{P\} \to X$ is a morphism of varieties, then we can extend f to $C \to X$.*

This follows by setting $Y = \mathrm{Spec}(k)$, $R = \mathscr{O}_{C,P}$ a DVR, and $K(R) = K(C)$ the function field of C.

Corollary 6.16. *1. Closed immersions are proper.*

2. Compositions of proper morphisms are proper.

3. Base extensions of proper maps are proper

4. Properness is determined locally. That is, $f : X \to Y$ a morphism it is proper iff for all $V \subseteq Y$ open subscheme, $f : f^{-1}(Y) \to Y$ is proper.

Exercise: Assume that P is a property of morphisms such that closed embeddings are P, compositions of P-morphisms are P and base extensions of P morphisms are P. Then products of P-morphisms are P, $X \xrightarrow{f} Y \xrightarrow{g} Z$, if gf is P and g is separated then f is P, and if $f : X \to Y$ is P then $f_{red} : X_{red} \to Y_{red}$ is P.

Projective Morphisms

$\mathbb{P}^n_A = \mathrm{Proj}\, A[x_0, \ldots, x_n] = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, A$.

Definition 6.26 (Projective Space over a Scheme)**.** *For any scheme Y, set $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times Y$*

Definition 6.27 (Projective Morphism)**.** *$f : X \to Y$ is projective if there exists a closed immersion $i : X \to \mathbb{P}^n_Y$ s.t. $f = \pi_Y \circ i$.*

f is quasi-projective if there is an open immersion $j : X \subset X'$ and a projective morphism $f' : X' \to Y$ s.y. $f = f' \circ j$.

Example: A variety X is quasi-projective iff $X \to \mathrm{Spec}\, k$ is quasiprojective, and likewise projective.

Example: S is a graded ring, gneerated bny finitely many elements of S_1 as an S_0-algebra. Then $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is projective.

Theorem 6.17. *A projective morphism of Nötherian schemes is proper.*

Proof. Enough to show that $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper.

Separated: Show that $\Delta : \mathbb{P}^n_{\mathbb{Z}} \to \mathbb{P}^n \times \mathbb{P}^n$ is a closed immbedding. $D_+(x_i x_j) \to D_+(x_i) \times D_+(x_j)$ gives $\mathbb{Z}[x_0, \dots, x_n]_{(x_i x_j)} \leftarrow \mathbb{Z}[x_0, \dots, x_n]_{(x_i)} \otimes_{\mathbb{Z}} \mathbb{Z}[x_0, \dots, x_n]_{(x_j)}$.

Properness; R a valuation ring, $K = K(R)$. We want γ such that $\pi\gamma = \alpha$ and $\gamma i = \beta$. Let $t_0 = \mathfrak{m}_R$, $t_1 = 0 \in \operatorname{Spec}(R)$. Set $p_1 = \beta(t_1) \in \mathbb{P}^n$. If $p_1 \in V(x_i)$, then induction imples that there is a $\gamma : \operatorname{Spec} R \to V(x_i) \simeq \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$.

WLOG: $p_1 \in D_+(x_0 x_1 \dots x_n) \Rightarrow x_i/x_j \in \mathscr{O}_{\mathbb{P},p_1}$ for all i, j.

$\beta^\sharp : k(p_1) \to K$, set $f_{ij} = \beta^\sharp(x_i/x_j) \in K$.

A valuation $v : K \to G$, $R = \{v \geq 0\}$. Choose m such that $v(f_{m0})$ is minimal. $v(f_{im}) = v(f_{i0}/f_{m0}) = v(f_{i0}) - v(f_{m0}) \geq 0$, so $f_{im} \in R$ for all i. So we have a ring homomorphism $\mathbb{Z}[x_0/x_m, \dots, x_n/x_m] \to R$ by $x_i/x_m \mapsto f_{im}$.

Thus, $\operatorname{Spec} R \xrightarrow{\gamma} D_+(x_m) \subseteq \mathbb{P}^n$ is as desired. □

Exercise:

1) There exists a closed Segre Embedding $\mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^m_{\mathbb{Z}} \to \mathbb{P}^{nm+n+m}_{\mathbb{Z}}$.

2) Composition of projective morphisms is projective.

Example: $X \overset{affine}{\to} Y \to \operatorname{Spec} k$, X a prevariety, Y a variety, then X is a variety.

$\mathscr{O}_X$ modules

X is a scheme, $\mathscr{F}, \mathscr{G}$ are sheaves of $\mathscr{O}_X$-modules.

$\hom_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) = \{$homomorphisms of $\mathscr{O}_X$-modules $\mathscr{F} \to \mathscr{G}\}$.

We define a sheaf of $\mathscr{O}_X$-modules $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ by $U \mapsto \hom_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$.

$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ by the sheafification of $U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$.

Let $f : X \to Y$ be morphisms of schemes, $\mathscr{H}$ an $\mathscr{O}_Y$-module. $f_*\mathscr{F}$ is an $f_*\mathscr{O}_X$-module, $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_Y$ a ring homomorphism implies that $f_*\mathscr{F}$ is an $\mathscr{O}_Y$-module.

The adjoint property: $\hom_{\mathscr{O}_Y}(\mathscr{H}, f_*\mathscr{F}) \iff \hom_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{H}, \mathscr{F})$.

Definition 6.28 (Pullback)**.** $f^*\mathscr{H} = f^{-1}\mathscr{H} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$*, this is an* $\mathscr{O}_X$*-module.*

Continued...see Sheaves from last semester

Proposition 6.18. $\tilde{M}_P = M_P$, $\tilde{M}(D(a)) = M_a$.

Corollary 6.19. $\Gamma(X, \tilde{M}) = M$ *and* $M \mapsto \tilde{M}$ *is an exact, fully faithful functor from A-mod to* $\mathscr{O}_X$*-mod with inverse the global section functor,* $\oplus_i \tilde{M}_i = \oplus_i \tilde{M}_i$ *and* $M \tilde{\otimes}_A N = \tilde{M} \otimes_{\mathscr{O}_X} \tilde{N}$

We have done this several times before, and so will not prove them again.

Proposition 6.20. $f : \operatorname{Spec} A = X \to \operatorname{Spec} B = Y$ *a morphism,* M *an* A*-module and* N *a* B*-module. Then* $f_*\tilde{M} = \tilde{M}_B$, $f^*\tilde{N} = N \tilde{\otimes}_B A$ *where* M_B *is* M *as a* B*-module using* $f^\sharp : B \to A$.

Proof. Let $b \in B$, $\Gamma(D(b), f_*\tilde{M}) = \Gamma(f^{-1}(D(b)), \tilde{M}) = \Gamma(D(f^\sharp(b)), \tilde{M}) = M_{f^\sharp(b)} = (M_B)_b$.

Let $P \in X$. $Q = (f^\sharp)^{-1}(P) = f(P) \in Y$. $(f^*\tilde{N})_P = \tilde{N}_{f(P)} \otimes_{\mathscr{O}_{Y,f(P)}} \mathscr{O}_{X,P} = N_Q \otimes_{B_Q} A_P = (N \otimes_B A)_P$. □

Definition 6.29 (Quasicoherent and Coherent Sheaves)**.** *Let X be a scheme. An $\mathscr{O}_X$-module $\mathscr{F}$ is quasi-coherent if there exists an open cover $X = \cup U_i$, $U_i = \operatorname{Spec} A_i$ and A_i modules M_i such that $\mathscr{F}|_{U_i} \simeq \tilde{M}_i$.*

$\mathscr{F}$ is coherent if the M_i are ginitely generated.

Example: $i : Y \to X$ a closed subscheme implies that $i_*\mathscr{O}_Y$ is a coherent $\mathscr{O}_X$-module. $U = \operatorname{Spec} A \subseteq X$ open, $i^{-1}(U) = \operatorname{Spec}(A/I)$, $i_*\mathscr{O}_Y = i_*(\tilde{A/I}) = \tilde{A/I}$.

Lemma 6.21. $X = \operatorname{Spec} A, f \in A$. *$\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.*

1. *If $s \in \Gamma(X, \mathscr{F})$ and $s|_{D(f)} = 0 \in \Gamma(D(f), \mathscr{F})$, then $\exists n > 0$ such that $f^n s = 0 \in \Gamma(X, \mathscr{F})$*
2. *If $t \in \Gamma(D(f), \mathscr{F})$ then $\exists s \in \Gamma(X, \mathscr{F})$ and $m > 0$ so that $s|_{D(f)} = f^n t$.*

Proof. $P \in X$, there exists an open nbhd $P \in U = \operatorname{Spec}(B) \subseteq X$ and B-module M such that $\mathscr{F}|_U \simeq \tilde{M}$. So there exists $g \in A$ such that $P \in D(g) \subseteq U$. $\mathscr{F}|_{D(g)} = \tilde{M}|_{D(g)} = M \tilde{\otimes}_B A_g$. Thus, we can write $X = D(g_1) \cup \ldots \cup D(g_m)$ such that $\mathscr{F}|_{D(g_i)} = \tilde{M}_i$, with M_i a A_{g_i}-module.

1. Let $s \in \Gamma(X, \mathscr{F})$, with $s|_{D(f)} = 0$. Set $m_i = s|_{D(g_i)} \in M_i$. Then $m_i|_{D(fg_i)} = 0 \in (M_i)_f$, so $f^n m_i = 0 \in M_i$ for all i, so $f^n s = 0 \in \Gamma(X, \mathscr{F})$.
2. $X = D(g_1) \cup \ldots \cup D(g_r)$. $\mathscr{F}|_{D(g_i)} = \tilde{M}_i$, where M_i is an A_{g_i}-module. $t|_{D(fg_i)} \in (M_i)_f$. $\exists t_i \in Mi$ such that $t_i|_{D(fg_i)} = f^n t|_{D(fg_i)}$. So then $(t_i - t_j)|_{D(fg_ig_j)} = (f^n t - f^n t)|_{D(fg_ig_j)} = 0$. By part 1, $f^m(t_i - t_j) = 0 \in \Gamma(D(g_ig_j), \mathscr{F})$. That is, $f^m t_i = f^m t_j$ on $D(g_i) \cap D(g_j)$. We flue $s \in \Gamma(X, \mathscr{F})$ s.t. $s|_{D(g_i)} = f^m t_i$ and $s|_{D(f)} = f^{n+m} t$, as $s|_{D(fg_i)} = f^m t_i = f^n f^m t|_{D(fg_i)}$.

□

Corollary 6.22. $\Gamma(D(f), \mathscr{F}) = \Gamma(X, \mathscr{F})_f$.

Proposition 6.23. *X a scheme, $\mathscr{F}$ an $\mathscr{O}_X$-module.*

1. *$\mathscr{F}$ is quasi-coherent iff for all open* $\operatorname{Spec} A = U \subseteq X$*, there exists an A-module M such that $\mathscr{F}|_U = \tilde{M}$*
2. *X Nötherian: $\mathscr{F}$ coherent iff the same is true but M finitely generated.*

Proof.
1. Assume that $\mathscr{F}$ is quasicoherent, $U = \operatorname{Spec} A \subseteq X$ is open. Then $\mathscr{F}|_U$ is a quasicoherent $\mathscr{O}_X$-module. Write $U = D(g_i) \cup \ldots \cup D(g_r)$, $g_i \in A$. $\mathscr{F}|_{D(g_i)} \simeq \tilde{M}_i$. Set $M = \mathscr{F}(U)$. $M \mapsto \Gamma(U, \mathscr{F})$ gives an $\mathscr{O}_U$-homomorphism $\alpha : \tilde{M} \to \mathscr{F}$. The lemma implies that $M_i = \mathscr{F}(D(g_i)) = M_{g_i}$. Therefore, α is an isomorphism over $D(g_i)$ implies that it is one over U.

2. Assume X Nötherian, $\mathscr{F}$ is coherent. Then A is Nötherian, so $M_i = M_{g_i}$ is a finitely generated module over A_{g_i} for all i. We must show that M is a finitely generated A-module: take $m_1, \ldots, m_N \in M$ such that M_{g_i} is gneerated by $\{m_j/1\}$ for all i. Then M is generated by $\{m_1, \ldots, m_N\}$ as an A-module.

□

Corollary 6.24. $X = \operatorname{Spec} A$. *Then there is an equivalence* $M \mapsto \tilde{M}$ *and* $\mathscr{F} \to \Gamma(X, \mathscr{F})$ *between A-modules and quasi-coherent $\mathscr{O}_X$-modules.*

Recall: $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is exact iff $0 \to \mathscr{F}'_p \to \mathscr{F}_p \to \mathscr{F}''_p \to 0$ is exact for all $p \in X$. We only get automatically that $0 \to \Gamma(U, \mathscr{F}') \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$ is exact.

Proposition 6.25. *X is affine scheme, $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ exact sequence, if $\mathscr{F}'$ is quasi-coherent, then $\Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$ is surjective, so $\Gamma(U, -)$ becomes an exact functor.*

Proposition 6.26. *Any kernel, cokernel or image of an $\mathscr{O}_X$-homomorphism of quasicoherent $\mathscr{O}_X$-modules is quasicoherent. An extension of quasi-coherent $\mathscr{O}_X$-modules is quasi-coherent.*

If X is Nötherian, then the same holds for coherent modules.

Proof. WLOG, $X = \operatorname{Spec} A$. $\varphi : \tilde{M} \to \tilde{N}$. This gives $0 \to K \to M \to N \to C \to 0$ an exact sequence, as $\tilde{}$ is exact, $0 \to \tilde{K} \to \tilde{M} \to \tilde{N} \to \tilde{C} \to 0$ is exact. So the kernel and cokernel must be quasicoherent. The image is $\operatorname{coker}(\tilde{K} \to \tilde{M})$, and so is also quasicoherent.

We now assume that $0 \to \tilde{M} \to \mathscr{F} \to \tilde{N} \to 0$ is a short exact sequence with $\tilde{M}, \tilde{N}$ quasicoherent and $\mathscr{F}$ an $\mathscr{O}_X$-module. Define $F = \Gamma(X, \mathscr{F})$. The proposition states that $0 \to M \to F \to N \to 0$ is a short exact sequence of A-modules. Then we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{F} & \longrightarrow & \tilde{N} & \longrightarrow & 0 \\ & & \downarrow\simeq & & \downarrow & & \downarrow\simeq & & \\ 0 & \longrightarrow & \tilde{M} & \longrightarrow & \mathscr{F} & \longrightarrow & \tilde{N} & \longrightarrow & 0 \end{array}$$

So by the five lemma, $\tilde{F} \simeq \mathscr{F}$. □

Definition 6.30 (Ideal Sheaf)**.** *Let X be a scheme, $i : Y \to X$ a closed immersion. Then the ideal sheaf of Y is $\mathscr{I}_Y = \ker(\mathscr{O}_X \xrightarrow{i^\sharp} i_*\mathscr{O}_Y) \subseteq \mathscr{O}_X$.*

Proposition 6.27. *{closed subschemes of X} correspond to {quasicoherent sheaves of ideals $\subseteq \mathscr{O}_X$}.*

Proof. Assume that $i : Y \to X$ is a closed immersion, then we have $0\mathscr{I}_Y \to \mathscr{O}_X \xrightarrow{i^\sharp} i_*\mathscr{O}_Y \to 0$. $i_*\mathscr{O}_Y$ is quasicoherent (we will prove this next, it is true because i is separated and quasicompact). Then $\mathscr{I}_Y$ is quasicoherent as well.

Assume that $\mathscr{I} \subseteq \mathscr{O}_X$ is quasicoherent. For $p \in X$, $\mathscr{I}_p \subseteq \mathscr{O}_{X,p}$. Define $Y = \{P \in X : \mathscr{I}_P \subseteq \mathfrak{m}_P \subsetneq \mathscr{O}_{X,P}\} = \operatorname{Supp}(\mathscr{O}_X/\mathscr{I}) = \{P \in X | (\mathscr{O}_X/\mathscr{I})_P \neq 0\}$. Let $i : Y \to X$ be the inclusion.

Define $\mathscr{O}_Y = i^{-1}(\mathscr{O}_X/\mathscr{I})$. Then $(Y, \mathscr{O}_Y)$ is a locally ringed space. $\mathscr{O}_{X,P} = (\mathscr{O}_X/\mathscr{I})_P = \mathscr{O}_{X,P}/\mathscr{I}_P$. Then $i^{-1}(\mathscr{O}_X/\mathscr{I}) \xrightarrow{\text{id}} \mathscr{O}_Y$, which corresponds to a map $\mathscr{O}_X/\mathscr{I} \xrightarrow{\varphi} i_*\mathscr{O}_Y$ by the adjoint property.

Then we define $i^\sharp : \mathscr{O}_X \to \mathscr{O}_X/\mathscr{I} \xrightarrow{\varphi} i_*\mathscr{O}_Y$, so $(i, i^\sharp) : Y \to X$ is a morphism of locally ringed spaces. Claim: This is a closed immersion.

WLOG $X = \operatorname{Spec} A$. Then $\mathscr{I} = \tilde{I} \subseteq \mathscr{O}_X$, with $I \subseteq A$ an ideal. Then $Y = V(I) \subseteq X$ is closed, and $i : Y \to X$ is a homeomorphism onto its image. We must check that $Y = \operatorname{Spec} A/I$.

$\mathscr{O}_{Y,P} = (\mathscr{O}_X/\mathscr{I})_P = A_P/I_P = (A/I)_P = \mathscr{O}_{\operatorname{Spec} A/I, P}$. □

Corollary 6.28. *All closed subschemes of* $\operatorname{Spec} A$ *are in correspondence with ideals of* A.

All closed subschemes of an affine scheme are affine.

Corollary 6.29. *If* $U_1, U_2 \subseteq X$ *open affine subschemes, and if* X *is separated, then* $U_1 \cap U_2$ *is affine.*

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ \uparrow & & \uparrow \subseteq \\ U_1 \cap U_2 & \longrightarrow & U_1 \times U_2 \end{array}$$

Proof. As Δ is a closed embedding, the bottom map must be as well. The product of affines over an affine is affine, and so then $U_1 \cap U_2$ is affine, as it is a subscheme of $U_1 \times U_2$. □

Proposition 6.30. *Let* $f : X \to Y$ *be a morphism.*

1. *$\mathscr{G}$ a quasi-coherent $\mathscr{O}_Y$-module, then $f^*\mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module.*
2. *X, Y both Nötherian, same for coherent.*
3. *Assume X is Nötherian or that f is separated and quasicompact, then if $\mathscr{F}$ is quasicoherent $\mathscr{O}_X$-module, then $f_*\mathscr{F}$ is a quasicoherent $\mathscr{O}_Y$-module*

Proof.
1. Let $P \in X$, Let $\operatorname{Spec} B = V \subseteq Y$ open affine such that $f(P) \in V$. Take $U = \operatorname{Spec} A \subseteq X$ open such that $P \in U \subseteq f^{-1}(V)$. $\mathscr{G}|_V = \tilde{M}$ with M a B-module. Then $(f^*\mathscr{G})|_U = \bar{f}^*(\mathscr{G}|_V) = \bar{f}^*(\tilde{M}) = M \tilde{\otimes}_B A$.
2. Similar, but noting finite generation everywhere.
3. WLOG, Y is affine. Last time, we showed that (c) is true when X is affine. Assumptions imply that there exists a finite open affine cover $X = \cup U_i$ such that $U_i \cap U_j$ has a finite open affine cover $U_i \cap U_j = \cup U_{ijk}$ finite. As $\mathscr{F}$ is a sheaf, we obtain an exact sequence of $\mathscr{O}_Y$-modules $0 \to f_*\mathscr{F} \to \oplus_i f_*\mathscr{F}|_{U_i} \to \oplus_{i,j,k} f_*\mathscr{F}|_{U_{ijk}}$. So $f_*\mathscr{F}$ is the kernel of a map of quasicoherent $\mathscr{O}_Y$-modules.

□

Previously, we've shown that the following properties are all equivalent

1. Closed subschemes of affine schemes are affine
2. If X is separated and $U_1, U_2 \subseteq X$ open affine, then $U_1 \cap U_2$ affine.
3. Pushforward of a quasicoherent sheaf is quasicoherent
4. The ideal sheaf of a closed subscheme is quasi-coherent.

We now prove number 1:

Proof. If $i : Y \to X$ is a homeomorphism onto $i(Y)$, and $i(Y)$ is closed, then $(i_*\mathscr{O}_Y)_P$ is $\mathscr{O}_{Y,P}$ if $P \in Y$ and 0 else.

For $f \in \Gamma(X, \mathscr{O}_X)$ write $X_f = D(f) = \{P \in X : f \notin \mathfrak{m}_P \subseteq \mathscr{O}_{X,P}\}$.

Assume that $i : Y \to X = \operatorname{Spec} A$ is a closed subscheme. Let $P \in Y$, then there exists open affine $V \subseteq Y$. Note: $\{Y_f = i^{-1}(X_f)\}$ is a basis for the topology on Y.

Thus, there exists $f \in A$ such that $P \in Y_f \subseteq V$, $Y_f = V_f$ is affine. So we can write $X = X_{f_1} \cup \ldots \cup X_{f_r}$ such that Y_{f_i} affine for all i. $(f_1, \ldots, f_r) = (1) \subseteq A$, thus $(i^\sharp(f_1), i^\sharp(f_2), \ldots, i^\sharp(f_r)) = (1) \subseteq \Gamma(Y, \mathscr{O}_Y)$.

By exercise 2.17b in Hartshorne, Y is affine iff $\exists f_1, \ldots, f_r \in \Gamma(Y, \mathscr{O}_Y)$ such that Y_{f_i} is affine for all i and $(f_1, \ldots, f_r) = 1 \subseteq \Gamma(Y, \mathscr{O}_Y)$. □

$S = \oplus_{d \geq 0} S_d$ a graded ring, $X = \operatorname{Proj}(S)$, and $M = \oplus_{d \in \mathbb{Z}} M_d$ a graded S-module. $S_d \cdot M_e \subseteq M_{e+d}$ for $P \in X$ defin an $S_{(P)}$-module $M_{(P)} = \{m/f : m \in M, f \in S \text{ homogeneous of the same degree}\}$.

For $U \subseteq X$ open, define $\mathscr{O}_X$-module $\tilde{M}$ by $\Gamma(U, \tilde{M}) = \{s : U \to \coprod_{P \in U} M_{(P)} | s(P) \in M_{(P)} \forall P \in U \text{ and } s \text{ is locally a quotient}\}$.

Proposition 6.31. *1.* $\tilde{M}_P = M_{(P)}$

2. $\tilde{M}|_{D_+(f)} = \tilde{M_{(f)}}$ *for* $f \in S_+$, $\deg(f) > 0$.

Corollary 6.32. *$\tilde{M}$ is quasicoherent. If S is Nötherian and M finitely generated S-module, then $\tilde{M}$ is coherent.*

Definition 6.31 (Twisted Modules)**.** *Define $M(n) = M$ as an S-module, but with grading $M(n)_e = M_{e+n}$.*

If $X = \operatorname{Proj}(S)$, define $\mathscr{O}_X(n) = \tilde{S(n)}$. For any $\mathscr{O}_X$-module, $\mathscr{F}$, define $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$.

Example:

1. $\mathscr{O}_X(0) = \tilde{S} = \mathscr{O}_X$.
2. $S = A[x_0, \ldots, x_m]$, $X = \operatorname{Proj}(S) = \mathbb{P}^m_A$. Then $\Gamma(D_+(x_i), \mathscr{O}_X(n)) = S(n)_{x_i} = \{f/x_i^r | f \in S_{r+n}\} = \{\text{elements of degree } n\} \subseteq S_{x_i}$

Claim: $\Gamma(X, \mathscr{O}_X(n)) = S_n$. We have a map $S_n = S(n)_0$, Let $t \in \Gamma(X, \mathscr{O}_X(n))$, set $t_i = t|_{D_+(x_i)} \in S_{x_i}$ element of degree n, if $t_i = t_j$ on $D_+(x_i x_j)$, then $t_i = t_j \in S_{x_i x_j}$, not that $S_{x_i} \subseteq S_{x_i x_j}$, $S_{x_i} \cap S_{x_j} = S \subseteq S_{x_i x_j}$, thus $t_i = t_j \in S_n$ for all i, j.

Proposition 6.33. *Let $X = \mathrm{Proj}(S)$, S generated by S_1 as an S_0-algebra.*

1. *$\mathscr{O}_X(n)$ is an invertible $\mathscr{O}_X$-module.*
2. *$\tilde{M} \otimes_{\mathscr{O}_X} \tilde{N} = M \tilde{\otimes}_S N$.*

Proof. 1. Let $f \in S_1$. $\Gamma(D_+(f), \mathscr{O}_X(U)) = S(n)_{(f)} \xrightarrow{\cdot f^{-n}} S_{(f)}$, which says that $\mathscr{O}_X(n)|_{D_+(f)} = \tilde{S(n)}_{(f)} \simeq \tilde{S}_{(f)} = \mathscr{O}_{D_+(f)}$.

2. $M \otimes_S N$ is a graded S-module, by $(M \otimes_S N)_e$ =submodule generated by $m \otimes n$ where $m \in M_r, n \in N_t$ with $r + t = e$. Let $P \in X$. Then we have a map $M_{(P)} \otimes_{S_{(P)}} N_{(P)} \to (M \otimes_S N)_{(P)}$ by $m/f \otimes n/g \mapsto m \otimes n/fg$. S is generated by S_1, so this is an isomorphism. Why? Because $P \not\supseteq S_+ \Rightarrow P \not\supseteq S_1 \Rightarrow \exists h \in S_1$ such that $h \notin P$. Let $m \otimes n/f \in (M \otimes N)_{(P)}$, go to $m/h^r \otimes h^r n/f \in M_{(P)} \otimes N_{(P)}$ where $m \in M_r$, $n \in N_t$ and $s \in S_{r+t}$.

   Construct an isomorphism $\varphi : \tilde{M} \otimes_{\mathscr{O}_X} \tilde{N} \to M \tilde{\otimes} N$ by $s \in \Gamma(U, \tilde{M} \otimes \tilde{N}) \mapsto [P \mapsto s_p \in \tilde{M}_P \otimes_{\mathscr{O}_{X,P}} \tilde{N}_P = (M \otimes N)_{(P)}]$, which is an isomorphism.

∎

Corollary 6.34. $\tilde{M}(n) = \tilde{M} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) = \tilde{M} \otimes \tilde{S(n)} = (M \otimes_S S(n))\tilde{} = \tilde{M(n)}$.

This says that $\mathscr{O}_X(m) \otimes \mathscr{O}_X(n) = \mathscr{O}_X(m+n)$.

Definition 6.32. *Let $X = \mathrm{Proj}(S)$, $\mathscr{F}$ an $\mathscr{O}_X$-module. Define $\Gamma_*(\mathscr{F}) = \oplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n))$. This is a graded S-module.*

Let $s \in S_d, t \in \Gamma(X, \mathscr{F}(n))$ gives $s \in \Gamma(X, \mathscr{O}_X(d))$, so $t \otimes s \in \Gamma(X, \mathscr{F}(n)) \otimes \mathscr{O}_X(d) = \Gamma(X, \mathscr{F}(n+d))$

Note: $X = \mathrm{Proj}\, S$ where $S = A[x_0, \ldots, x_n]$. $\Gamma_*(\mathscr{O}_X) = S$. This is NOT always true!

(Side Note: Look at the operation $\Gamma_*(\mathrm{Proj}\, S)$. Is it a functor? What properties does it have? Filling in lower degrees?)

Let X be a scheme, $\mathscr{I}$ an invertible $\mathscr{L}_X$-module, and $f \in \Gamma(X, \mathscr{L})$.

Definition 6.33. *$X_f = \{P \in X | f_P \notin \mathfrak{m}_P \mathscr{L}_X\}$. $X_f \subseteq X$ is open.*

Lemma 6.35. *Let X be a quasi-compact scheme, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, $f \in \Gamma(X, \mathscr{L})$ and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.*

1. *Let $s \in \Gamma(X, \mathscr{F})$, $s|_{X_f} = 0$, then $s \otimes f^n = 0 \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$.*

Proof. 1. $X = U_1 \cup \ldots \cup U_r$ where $U_i = \mathrm{Spec}\, A_i$ and $\mathscr{L}|_{U_i} \simeq \mathscr{O}_{U_i}$. $\mathscr{F}|_{U_i} = \tilde{M}_i$ with M_i an A_i-module. $s_i|_{U_i} \in M_i$. $f_i = f|_{U_i} \in \Gamma(U_i, \mathscr{L}) \simeq A_i$, $\mathscr{F} \otimes \mathscr{L}^{\otimes n}|_{U_i} \simeq \tilde{M}_i \otimes \mathscr{O}_{U_i}^{\otimes n} = \tilde{M}_i$ and $s \otimes f^n \mapsto f_i^n s_i = 0$ for $n >> 0$, so $s \otimes f^n = 0 \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$.

∎

X a schemem $\mathscr{L}$ an invertible $\mathscr{O}_X$-module.

Definition 6.34. $\mathscr{L}^{-1} = \mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$.

If $\mathscr{L}|_U \simeq \mathscr{O}_U$ then $\mathscr{L}^{-1}|_U \simeq \mathscr{H}om(\mathscr{O}_U, \mathscr{O}_U) \simeq \mathscr{O}_U$.

Note: Global $\mathscr{O}_X$-hom: $\mathscr{L} \otimes \mathscr{L}^{-1} \to \mathscr{O}_X$ by $s \otimes \alpha \mapsto \alpha(s)$ is an isomorphism on U.

Let $f \in \Gamma(X, \mathscr{L})$, $X_f = \{P \in X : f_P \notin \mathfrak{m}_P \mathscr{L}_P\} \subseteq X$ open. $\mathscr{L}^{-1}|_{X_f} \to \mathscr{O}_{X_f}$, $\alpha \mapsto \alpha(f)$.

Define $f^{-1} \in \Gamma(X_f, \mathscr{L}^{-1})$ to be the inverse image of 1 by this isomorphism.

So $f \otimes f^{-1} = 1$, adn $\mathscr{L} \otimes \mathscr{L}^{-1} \simeq \mathscr{O}_X$.

Lemma 6.36. *Let X be a quasi-compact scheme, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, $f \in \Gamma(X, \mathscr{L})$ and $\mathscr{F}$ a quasicoherent $\mathscr{O}_X$-module.*

1. *$s \in \Gamma(X, \mathscr{F})$, $s|_{X_f} = 0 \Rightarrow s \otimes f^n = 0 \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$*

2. *Assume $X = U_1 \cup \ldots \cup U_r$ with U_i open affine and $\mathscr{L}|_{U_i} \simeq \mathscr{O}_{U_i}$ for all i, and $U_i \cap U_j$ quasicompact. $t \in \Gamma(X_f, \mathscr{F}) \Rightarrow \exists s \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$ such that $s|_{X_f} = t \otimes f^n$.*

Corollary 6.37. *$S = \oplus_{d \geq 0} \Gamma(X, \mathscr{L}^{\otimes d})$ is a graded ring, $f \in S_1$. $\Gamma_* \mathscr{F} = \oplus_{e \in \mathbb{Z}} \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes e})$ is a graded Smodule. Then $(\Gamma_* \mathscr{F})_{(f)} = \Gamma(X_f, \mathscr{F})$*

Proof. $(\Gamma_* \mathscr{F})_{(f)} = \{s/f^d | s \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes d})\}$

$f^{-d} \in \Gamma(X_f, \mathscr{L}^{\otimes -d})$. $(\Gamma_* \mathscr{F})_{(f)} \to \Gamma(X_f, \mathscr{F})$ by $s/f^d \mapsto s \otimes f^{-d}$. This is injective and surfective. □

Let $S = \oplus_{d \geq 0} S_d$ be a graded ring, $X = \operatorname{Proj} S$.

Recall, $\mathscr{O}_X(n) = \widetilde{S(n)}$. $f \in S_d$ implies $f \in S(d)_0 \Rightarrow f \in \Gamma(X, \mathscr{O}_X(d))$. $\mathscr{F}$ an $\mathscr{O}_X$-module, $\Gamma_* \mathscr{F} = \oplus_{e \in \mathbb{Z}} \Gamma(X, \mathscr{F}(e))$. Construct an $\mathscr{O}_X$-hom $\beta : \widetilde{\Gamma_* \mathscr{F}} \to \mathscr{F}$ by $\beta|_{X_f} : \widetilde{\Gamma_* \mathscr{F}}_{(f)} \to \mathscr{F}|_{X_f}$, which corresponds to a module homomorphism $(\Gamma_* \mathscr{F})_{(f)} \to \Gamma(X_f, \mathscr{F})$ by $s/f^m \mapsto s \otimes f^{-m}$.

Proposition 6.38. *If $\mathscr{F}$ is quasi-coherent, and S is finitely generated by S_1 as an S_0-algebra, then $\beta : \widetilde{\Gamma_* \mathscr{F}} \to \mathscr{F}$ is an isomorphism.*

Proof. Assume $S_0[f_1, \ldots, f_r] \to S$ is surjective, $f_i \in S$. $U_i = X_{f_i}$ affine. $X = U_1 \cup \ldots \cup U_r$. $\mathscr{O}(1)|_{U_i} = \widetilde{S(1)_{(x_i)}} = \widetilde{S_{(x_i)}} = \mathscr{O}_{U_i}$, $U_i \cap U_j = X_{f_i f_j}$ affine, so the lemma implies that $\beta|_{X_{f_i}}$ is an isomorphism for all i.

Let $I \subseteq S$ be a homogeneous ideal. Natural inclusion $i : Y = \operatorname{Proj}(S/I) \to X = \operatorname{Proj}(S)$ $i(Y) = V(I)$, $i_* \mathscr{O}_Y = \widetilde{S/I}$ (exercise)

So $\mathscr{O}_X \to \tilde{S} \to \widetilde{S/I} = i_* \mathscr{O}_Y$, so $Y \subseteq X$ is a closed subscheme. Note that $\mathscr{I}_Y = \tilde{I} \subseteq \mathscr{O}_X$. □

Corollary 6.39.
1. *Assume $Y \subseteq \mathbb{P}^r_A$ a closed subscheme with $\mathbb{P}^r_A = \operatorname{Proj} S$, $S = A[x_0, \ldots, x_r]$. Then there exists a homogeneous ideal $I \subseteq S$ such that $Y = \operatorname{Proj}(S/I)$.*

2. *A morphism* $\varphi : Y \to \operatorname{Spec} A$ *is projective iff* $Y = \operatorname{Proj}(S), S_0 = A$ *and* S *is finitely generated by* S_1*.*

Proof. 1. $\mathscr{I}_Y \subseteq \mathscr{O}_{\mathbb{P}^r_A}$ a quasicoherent subsheaf, $\mathscr{I}_Y \otimes \mathscr{O}(d) \subseteq \mathscr{O}(d)$ implies that $I = \Gamma_* \mathscr{I}_Y \subseteq \Gamma_* \mathscr{O}_{\mathbb{P}^r_A} = S$ a homogeneous ideal.

   $\operatorname{Proj}(S/I) \subseteq \mathbb{P}^r_A$ has ideal sheaf $\tilde{I} = \widetilde{\Gamma_* \mathscr{I}_Y} = \mathscr{I}_Y \subseteq \mathscr{O}_{\mathbb{P}^r_A}$. Thus, $Y = \operatorname{Proj}(S/I)$.

2. φ is projective iff it factors through $\mathbb{P}^r_A$ as a closed immersion for some r iff $Y = \operatorname{Proj} S$, with $S = A[x_0, \dots, x_r]/I$.

□

Definition 6.35 (Twisting Sheaf)**.** *Let* Y *be any scheme. The twisting sheaf of* $\mathbb{P}^r_Y = \mathbb{P}^r_{\mathbb{Z}} \times Y \xrightarrow{\pi} \mathbb{P}^r_{\mathbb{Z}}$ *is* $\mathscr{O}(1) = \pi^* \mathscr{O}_{\mathbb{P}^r}(1)$*.*

Definition 6.36 (Immersion)**.** *A morphism* $i : X \to Z$ *is an immersion if we can factor it as* $i : X \to Z_1 \to Z$ *with* $X \to Z_1$ *an open immersion and* $Z_1 \to Z$ *a closed immersion.*

Exercise: A composition of immersions is an immersion.

Definition 6.37 (Very Ample)**.** *Let* X *be a scheme over* Y *and* $\mathscr{L}$ *an invertible* $\mathscr{O}_X$*-module.* $\mathscr{L}$ *is very ample relative to* Y *if* $\exists$ *an immersion* $i : X \to \mathbb{P}^r_Y$ *such that* $\mathscr{L} = i^* \mathscr{O}(1)$*.*

Note: If $\varphi : X \to Y$ is projective, then φ is proper, and X has a very ample invertible sheaf relative to Y.

Suppose that $\varphi : X \to Y$ is proper and X has a very ample invertible $\mathscr{L}$ relative to Y. We have an immersion $i : X \to \mathbb{P}^r_Y$, φ proper implies that i is proper, so $i(X) \subseteq \mathbb{P}^r_Y$ is closed, and so i is a closed immersion.

Definition 6.38 (Generated by global sections)**.** *Let* X *be a scheme,* $\mathscr{F}$ *an* $\mathscr{O}_X$*-module,* $\mathscr{F}$ *is generated by globa section if* $\exists \oplus_{i \in I} \mathscr{O}_X \xrightarrow{\alpha} \mathscr{F}$ *a surjective* $\mathscr{O}_X$*-hom.*

IE, $\exists s_i \in \Gamma(X, \mathscr{F})$ *such that* $\mathscr{F}_P$ *is generayed by* $\{(s_i)_P\}$ *as an* $\mathscr{O}_{X,P}$*-module for all* $P \in X$*.*

Examples:

1. $X = \operatorname{Spec} A$, $\mathscr{F} = \tilde{M}$.

2. $X = \operatorname{Proj} S$, S generated by S_1 as an S_0-module. Then $\mathscr{O}_X(1)$ is generated by global sections.

Definition 6.39 (Ample Sheaf)**.** *An invertible* $\mathscr{O}_X$*-module* $\mathscr{L}$ *on a Nötherian scheme* X *is ample if* $\forall$ *coherent* $\mathscr{O}_X$*-modules* $\mathscr{F}$*,* $\exists n_0 > 0$ *such that* $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ *generatd by global sections for all* $n \geq n_0$*.*

Later: If X is of finite type over $\operatorname{Spec} A$, A Nötherian, then $\mathscr{L}$ is ample iff $\mathscr{L}^{\otimes m}$ is very ample relative to A for some $m > 0$.

Theorem 6.40. *Let X be projective over* $\operatorname{Spec} A$*, A Nötherian. If $\mathscr{O}(1)$ is very ample relative to A, then $\mathscr{O}(1)$ is ample.*

Proof. $i : X \to \mathbb{P}^r_A$ is a closed immersion, $\mathscr{O}(1) = i^*\mathscr{O}_{\mathbb{P}^r}(1)$. $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Then $i_*\mathscr{F}$ is coherent on $\mathbb{P}^r_A$. For $P \in X$, $(i_*\mathscr{F})_P = \mathscr{F}_P$.

Exercise: $\Gamma(X, \mathscr{F} \otimes \mathscr{O}(1)^{\otimes m}) = \Gamma(\mathbb{P}^r_A, (i_*\mathscr{F})(m))$.

WLOG: $X = \mathbb{P}^r_A = \operatorname{Proj} A[x_0, \ldots, x_r]$.

$\mathscr{F}$ coherent implies that $\mathscr{F}|_{D_+(x_i)} \simeq \tilde{M}_i$, with M_i a finitely generated B_i module, where $B_i = A[x_0/x_i, \ldots, x_r/x_i]$. Let $s_{i1}, \ldots, s_{iN}$ generate M_i as a B_i-module. Then the lemma implies that there exists $t_{ij} \in \Gamma(X, \mathscr{F} \otimes \mathscr{O}_X(n))$ such that $t_{ij}|_{D_+(x_i)} = s_{ij} \otimes x_i^n$.

Take n large enough for all i, j. Claim: $\mathscr{F}(n) = \mathscr{F} \otimes \mathscr{O}_X(n)$ is generated by $\{t_{ij}\} \subset \Gamma(X, \mathscr{F}(n))$.

Why? Because $\mathscr{F}|_{D_+(x_i)} \xrightarrow{\simeq} \mathscr{F}(n)|_{D_+(x_i)}$ by $s \mapsto s \otimes x_i^n$. □

Corollary 6.41. *X projective over Nötherian A, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Then there exists $\oplus\mathscr{O}_X(n_i) \to \mathscr{F}$ surjective with a finite sum.*

Proof. The theorem says that $\mathscr{F} \otimes \mathscr{O}(n) = \mathscr{F}(n)$ is generated by global sections. Thus, $\mathscr{O}_X^{\oplus N} \to \mathscr{F}(U) \to 0$ exact, tensor with $\mathscr{O}(-n)$, and we get the result. □

Definition 6.40 (*-scheme)**.** *X is a *-scheme if it is Nötherian, separated, integral and $\mathscr{O}_{X,P}$ is a DVR whenever* $\dim \mathscr{O}_{X,P} = 1$.

Examples: nonsing alg variety

X normal, Nötherian, separated and integral.

Definition 6.41 (Prime Divisor)**.** *Assume X is a *-scheme, a prime divisor is a closed integral subscheme of codimension 1.*

Note that $\mathscr{O}_{X,Y}$ is a DVR, we have a valuation $v_Y : k(X)^* \to \mathbb{Z}$ by $\mathscr{O}_{X,Y} = \{f \in k(X) | v_Y(f) \geq 0\}$. We call $v_Y(f)$ the order of vanishing of f along Y. $v_(fg) = v_Y(f) + v_Y(g)$ and $v_Y(f+g) \geq \min(v_Y(f), v_Y(g))$.

We set $\operatorname{Div}(X)$ to be the free abelian group on the prime divisors. A principal divisor is one of the form $(f) = \sum_Y v_Y(f)[Y]$ with $f \in k(X)^*$. For $D, D' \in \operatorname{Div}(X)$ write $D \sim D' \iff D - D'$ is principle.

Define the divisor class group $\mathrm{C}\ell(X) = \operatorname{Div}(X)/\{\text{principal divisors}\}$.

Example: $X = \mathbb{P}^n_k$, a prime divisor is $Y = V(f)$ and $\deg(Y) = \deg(f)$. There is an ismorphism $\mathrm{C}\ell(\mathbb{P}^n) \to \mathbb{Z}$ by $\sum n_i[Y_i] \mapsto \sum n_i \deg(Y_i)$.

1. $U \subseteq X$ open implies there exists a surjective group hom $\mathrm{C}\ell(X) \to \mathrm{C}\ell(U)$ by $[Y] \mapsto [Y \cap U]$ if $Y \cap U \neq \emptyset$ and 0 otherwise. So if $Z = X \setminus U$ and $\operatorname{codim}(Z; X) \geq 2$, then $\mathrm{C}\ell(X) \simeq \mathrm{C}\ell(U)$. If $Z \subseteq X$ is a prime divisor, then have $\mathbb{Z} \to \mathrm{C}\ell(X) \to \mathrm{C}\ell(U) \to 0$.

2. A is a Nötherian domain. Then A is a UFD iff $U = \operatorname{Spec} A$ is normal and $\mathrm{C}\ell(U) = 0$.

3. $\pi : X \times \mathbb{A}^m \to X$ gives an isomorphism $\pi^* : \mathrm{C}\ell(X) \to \mathrm{C}\ell(X \times \mathbb{A}^m)$ by $[Y] \mapsto [Y \times \mathbb{A}^m]$.

Example: $X = \mathbb{P}^n \times \mathbb{P}^m$, $p : X \to \mathbb{P}^n$ and $q : X \to \mathbb{P}^m$, then $p^* : \mathrm{C}\ell(\mathbb{P}^n) \to \mathrm{C}\ell(X)$ is injective, by $[Y] \mapsto [Y \times \mathbb{P}^m]$. Let $H \subseteq \mathbb{P}^m$ be a hyperplane, $U = X \setminus \mathbb{P}^n \times H = \mathbb{P}^n \times \mathbb{A}^m$. Then we have an isomorphism $\mathrm{C}\ell(\mathbb{P}^n) \to \mathrm{C}\ell(X) \to \mathrm{C}\ell(\mathbb{P}^n \times \mathbb{A}^n)$. Similarly $q^* : \mathrm{C}\ell(\mathbb{P}^m) \to \mathrm{C}\ell(X)$ is injective. In fact, the image of q^* is $\mathbb{Z}[\mathbb{P}^n \times H]$.

So we get $0 \to \mathrm{C}\ell(\mathbb{P}^m) = \mathbb{Z} \to \mathrm{C}\ell(X) \to \mathrm{C}\ell(\mathbb{P}^n \times \mathbb{A}^m) = \mathbb{Z} \to 0$, but the last map has a section, so we get $\mathrm{C}\ell(X) = \mathbb{Z}[\mathbb{P}^n \times H] \oplus \mathbb{Z}[H' \times \mathbb{P}^m]$.

Carier Divisors

Let X be any scheme. For $U \subseteq X$ open, let $S(U) \subseteq \mathscr{O}_X(U)$ be the set of non-zerodivisors. We define $\mathscr{K}$ to be the sheafification of $pre-\mathscr{K}$, which has $pre-\mathscr{K}(U) = S(U)^{-1}\mathscr{O}_X(U)$. This is a sheaf of rings, $\mathscr{O}_X$-module, $\mathscr{O}_X \to \mathscr{K}$.

Example: if X is an integral scheme, $S(U) = \mathscr{O}_X(U)\setminus\{0\}$. Then for U affine, $\Gamma(U, pre-\mathscr{K}) = K(\mathscr{O}_X(U)) = k(U)$ and $\Gamma(U, \mathscr{K}) = k(X)$ for any nonempty $U \subseteq X$ open.

Note that $\mathscr{O}_X \subseteq pre-\mathscr{K}$ is a sub-presheaf, so $pre-\mathscr{K}$ is a decent presheaf. Recall that a decent presheaf satisfies the first sheaf axiom, and $\mathscr{O}_X \subset pre-\mathscr{K} \subset \mathscr{K}$.

Define $\mathscr{K}^*(U)$ to be the set of invertible elements of $\mathscr{K}(U)$. This is a sheaf of abelian groups. So $\mathscr{O}_X^* \subseteq \mathscr{K}^*$.

Definition 6.42 (Cartier Divisor)**.** *A Cartier Divisor is an element* $D \in \Gamma(X, \mathscr{K}^*/\mathscr{O}^*)$. *It is principleif it is in the image of* $\Gamma(X, \mathscr{K}^*) \to \Gamma(X, \mathscr{K}^*/\mathscr{O}^*)$.

NOTE: This is not surjective! There is a sheafification involved.

Convention: We use additive notation.

Let D be a Cartier Divisor. Then there exists an open cover $X = \cup U_i$ and $f_i \in \Gamma(U_i, \mathscr{K}^*)$ such that $D|_{U_i}$ is the image of f_i and on $U_i \cap U_j$, $f_i/f_j \in \Gamma(U_i \cap U_j, \mathscr{O}_X^*)$.

Whenever $\{f_i\}$ satisfy this condition, they define a Cartier divisor D.

Definition 6.43. *We define* $\mathrm{CaC}\ell(X) = \{$*cartier divisors*$\}/\{$*principal cartier divisors*$\}$. *Write* $D \sim D'$ *iff* $D - D' = 0 \in \mathrm{CaC}\ell(X)$.

Assume that X is a *-scheme. Then D is a Cartier, $Y \subseteq X$ a prime divisor, we write $v_Y(D) = v_Y(f_i)$ where $Y \cap U_i \neq \emptyset$ and $D|_{U_i}$ =image of d_i. This gives us a group homomorphism $\Gamma(X, \mathscr{K}^*/\mathscr{O}_X^*) \to \mathrm{Div}(X)$ by $D \mapsto \sum_Y v_Y(D)[Y]$.

This induces a map $\mathrm{CaC}\ell(X) \to \mathrm{C}\ell(X)$, and is injective when X is normal.

Definition 6.44 (Locally Factorial)**.** *X is locally factorial if $\mathscr{O}_{X,P}$ is a UFD for all $P \in X$.*

Proposition 6.42. *X is locally factorial *-scheme, then* $\mathrm{CaC}\ell(X) \simeq \mathrm{C}\ell(X)$.

Definition 6.45 (Picard Group)**.** *The set of all isomorphism classes of invertible $\mathscr{O}_X$-modules under tensor product.*

Let $D \in \Gamma(X, \mathscr{K}^*/\mathscr{O}_X^*)$ cartier. Then $D|_{U_i}$ is the image of $f_i \in \Gamma(U, \mathscr{K}^*)$.

Definition 6.46. *$\mathscr{L}(D) \subseteq \mathscr{K}$ is the sub $\mathscr{O}_X$-module such that $\mathscr{L}(D)|_{U_i}$ is generated by f_i^{-1}.*

$\mathscr{L}(D)$ is an invertible $\mathscr{O}_X$-mod as $\mathscr{O}_{U_i} \to \mathscr{L}(D)|_{U_i}$ given by multiplication by f_i^{-1}.

Proposition 6.43. *1.* $\Gamma(X, \mathscr{K}^*/\mathscr{O}_X^*) \leftrightarrow \{$*invertible subsheafs of* $\mathscr{L}\}$.

2. $\mathscr{L}(D_1) \otimes \mathscr{L}(D_2)^{-1} \simeq \mathscr{L}(D_1 - D_2)$.

3. $D_1 \sim D_2$ *iff* $\mathscr{L}(D_1) \simeq \mathscr{L}(D_2)$.

Proof. 1. Let $\mathscr{L} \subseteq \mathscr{K}$ be any invertible $\mathscr{O}_X$-module. There exists an open cover $X = \cup U_i$ such that $\mathscr{L}|_{U_i} \simeq \mathscr{O}_{U_i}$, so $\Gamma(U_i, \mathscr{L}) \simeq \Gamma(U_i, \mathscr{O}_{U_i})$. Define f_i to be the section corresponding to $1 \in \Gamma(U_i, \mathscr{O}_{U_i})$ by the chosen isomorphism. $f_i/f_j \in \Gamma(U_i \cap U_j, \mathscr{O}_X^*) \Rightarrow \{f_i^{-1}\}$ define a Cartier divisor D.

2. $\mathscr{L}(D_1) \otimes \mathscr{L}(D_2)^{-1} \to \mathscr{L}(D_1 - D_2)$ by $h_1 \otimes h_2 \mapsto h_1 h_2$.

3. Assume $\mathscr{L}(D_1) \simeq \mathscr{L}(D_2)$ as $\mathscr{O}_X$-modules. By the last part, $\mathscr{K} \subset \mathscr{L}(D_1 - D_2) \simeq \mathscr{O}_X$. Let $1 \in \Gamma(X, \mathscr{O}_X)$ which corresponds to $f \in \Gamma(X, \mathscr{K})$. $f \in \Gamma(X, \mathscr{K}^*)$, so $D_1 - D_2 =$image of f^{-1}.

□

Corollary 6.44. *If X is any scheme, then the map* $\mathrm{CaC}\ell(X) \to \mathrm{Pic}(X)$ *is injective.*

Proposition 6.45. *If X is integral, then* $\mathrm{CaC}\ell(X) \simeq \mathrm{Pic}(X)$.

Proof. Must show that any invertible $\mathscr{L}$ is a submodule of $\mathscr{K}$.

Let $\mathscr{L}|_U \simeq \mathscr{O}_U$. Then $1 \in \Gamma(U, \mathscr{O}_U)$ corresponds to $f \in \Gamma(U, \mathscr{L})$. Assume that $\emptyset \neq V \subseteq X$ open, $h \in \mathscr{L}(V)$, then $U \cap V \neq \emptyset$. $h \otimes f^{-1} \in \Gamma(U \cap V, \mathscr{L} \otimes \mathscr{L}^{-1}) = \Gamma(U \cap V, \mathscr{O}_X) \subseteq \Gamma(U \cap V, \mathscr{K}) = \Gamma(V, \mathscr{K})$.

Define an $\mathscr{O}_X$-hom $\mathscr{L} \to \mathscr{K}$ by $h \mapsto h \otimes f^{-1}$. This is injective, as f_p generator for $\mathscr{L}_p$ for all $p \in U$. □

Corollary 6.46. *X Nötherian, Integral, Separated, Locally Factorial implies* $\mathrm{C}\ell(X) = \mathrm{CaC}\ell(X) = \mathrm{Pic}(X)$

Corollary 6.47. $\mathrm{Pic}(\mathbb{P}^n_k) = \{\mathscr{O}(m) | m \in \mathbb{Z}\}$.

Proof. $X = \mathbb{P}^n_k$. We know that $\mathrm{C}\ell(X) = \mathbb{Z}[H] \simeq \mathrm{Pic}(X)$.

Must show that $[H] \leftrightarrow \mathscr{O}(1)$. $H = V(h)$ where $h \in k[x_0, \ldots, x_n]$ is a linear form. Set $f_i = h/x_i \in \mathscr{O}_X(D_+(x_i))$. $f_i/f_j = x_j/x_i$ is a unit on $D_+(x_i x_j)$, so $\{f_i\}$ defines a Cartier Divisor. $D \in CaCl(X)$ thar corresponds to $[H]$ since $v_Y(D) = 0$ if $Y \neq H$ and 1 if $Y = H$. So we get $\mathscr{O}_X(1) \to \mathscr{L}(D)$ an isomorphism, by $s \mapsto s/h$. □

Definition 6.47 (Effective Divisor)**.** *A Cartier Divisor D is effective if $\exists$ an open cover $X = \cup U_i$ and nonzerodivisors $f_i \in \mathscr{O}_X(U_i)$ such that $D|_{U_i}$ is the image of f_i.*

D is effective gives us a closed subscheme of X by $\mathscr{I}_D$ generated by f_i on U. This gives us a correspondence between effective Cartier Divisors and closed subschemes that are locally generated by one nonzerodivisor.

Note: If X is a *-scheme and D an effectve Cartier Divisor, then $v_Y(D) = v_Y(f_i) \geq 0$ so $\sum_Y v_Y(D)[Y]$ is effective. If X is a locally factorial *-scheme, then effective Cartier divisors correspond to effective Weil divisors.

Commutative Algebra Fact: If A is a normal Nötherian Domain, then $A = \cap_{\mathrm{ht}(P)=1} A_P \subseteq K(A)$. So $f_i \in k(U_i)$ gives $f_i \in \mathscr{O}_{U_i}(U_i) \iff (f_i) \in \mathrm{Div}(U_i)$ effective.

Proposition 6.48. *$D \subseteq X$ an effective Cartier divisor implies that $\mathscr{I}_D = \mathscr{L}(-D) \subseteq \mathscr{K}$.*

This is because $\mathscr{O}_D \subseteq \mathscr{O}_X \subseteq \mathscr{K}$ is locally generated by f_i.

Let $\varphi : X \to Y$, $f \in \mathscr{O}_Y(Y)$ then $\varphi^{-1}(Y_f) = X_{\varphi^\sharp f}$.

In fact, we can do this for any line bundle $\mathscr{L}$. Let $\mathscr{L}$ be an $\mathscr{O}_Y$-module. Then $\varphi^* = \varphi^{-1}\mathscr{L} \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X$. For $s \in \Gamma(Y, \mathscr{L})$ define $\varphi^* s = \varphi^{-1}s \otimes 1 \in \Gamma(X, \varphi^*\mathscr{L})$.

$X_{\varphi^* s} = \{P \in X | (\varphi^* s)_P \notin \mathfrak{m}_P(\varphi^*\mathscr{L})_P\} = \varphi^{-1}(Y_s)$.

Morphisms to $\mathbb{P}^n$.

If $\varphi : X \to \mathbb{P}^n_A$ is a morphism, then

1. X is a scheme over A
2. $\mathscr{L} = \varphi^*\mathscr{O}(1)$ is an invertible $\mathscr{O}_X$-module.
3. Generated by $s_i = \varphi^* x_i \in \Gamma(X, \mathscr{L})$ for $0 \leq i \leq n$ where $\mathscr{O}(1)$ is generated by $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathscr{O}(1))$.

Claim: 1+2+3 determines a unique $\varphi : X \to \mathbb{P}^n_A$.

Note: $s_i^{-1} \in \Gamma(X_{s_i}, \mathscr{L}^{-1})$. $s_j/s_i = s_j \otimes s_i^{-1} \in \Gamma(X_{s_i}, \mathscr{L} \otimes \mathscr{L}^{-1}) = \Gamma(X_{s_i}, \mathscr{O}_X)$.

We must have $X_{s_i} = \varphi^{-1}(D_+(x_i))$. This defines $\varphi : X_{s_i} \to D_+(x_i)$ by A-algebra homomorphism $A[x_0/x_i, \ldots, x_n/x_i] \to \Gamma(X_{s_i}, \mathscr{O}_X)$ by $x_j/x_i \mapsto s_j/s_i$.

Theorem 6.49. *Let X be a scheme over A. Then there is a correspondence $\{\varphi : X \to \mathbb{P}^n_A$ over $A\}$ to $\{(\mathscr{L}, s_0, \ldots, s_n) | \mathscr{L}$ generated by $s_0, \ldots, s_n\}/\simeq$.*

Example: if k is a field, then $\mathbb{P}^n_k = \mathbb{A}^n_k \cup \mathbb{P}^{n-1}_k$, $\mathbb{A}^n = D_+(x_0)$ and $P \in \mathbb{P}^n$. Then $P = 0 \in \mathbb{A}^n$. So $\pi : \mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1}$, $\mathrm{id} : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ defines the projection from a point $\mathbb{P}^n \setminus \{P\} \to \mathbb{P}^{n-1}$.

Let $x_1, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathscr{O}(1))$. These generate $\mathscr{O}(1)|_{\mathbb{P}^n \setminus \{P\}}$. This defines the projection from P.

Note: Assume $\varphi : X \to \mathbb{P}^n_A$ is given by $s_i = \varphi^* x_i \in \mathscr{L} = \varphi^*\mathscr{O}(1)$. φ is a closed immersion iff $\varphi : X_{s_i} \to D_+(x_i)$ is a closed immersion for all i iff X_{s_i} affine and $A[x_0/x_i, \ldots, x_n/x_i] \to \mathscr{O}_X(X_{s_i})$ is surjective.

Recall the definition of an ample line bundle.

Example: If X is projective over a Nötherian ring, $\mathscr{L} = \mathscr{O}(1)$ is very ample. If X is affine, and $\mathscr{L}$ is any invertible sheaf.

Proposition 6.50. *X a Nötherian scheme, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. Then TFAE*

1. *$\mathscr{L}$ ample*
2. *$\mathscr{L}^{\otimes m}$ ample for all $m \geq 1$*
3. *$\mathscr{L}^{\otimes m}$ ample for some $m \geq 1$.*

Proof. $1 \Rightarrow 2 \Rightarrow 3$: trivial

Assume that $\mathscr{L}^{\otimes m}$ is ample, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. $\mathscr{F} \otimes \mathscr{L}^{\otimes k}$ is coherent. For $0 \leq k \leq m-1$ choose $n_k > 0$ such that $(\mathscr{F} \otimes \mathscr{L}^{\otimes k}) \otimes (\mathscr{L}^{\otimes m})^{\otimes n}$ is generatedby global sections for all $n \geq n_k$.

$N = \max\{k + nm_k\}$ implies that $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections for all $n \geq N$. □

Theorem 6.51. *X is of finite type over Nötherian A. Then $\mathscr{L}$ is ample iff $\mathscr{L}^{\otimes m}$ isvery amples relative to* $\operatorname{Spec} A$ *for some $m > 0$.*

$\mathbb{P}^n_k$: $\mathscr{O}(m)$ is ample iff $m \geq 1$. If $m < 0$ then $\Gamma(\mathbb{P}^n, \mathscr{O}(m)) = 0$. $\mathscr{O}(0)$ gen by global sections.

Remark: $\varphi : X \to Y$ a morphism, (X Nötherian OR φ separated and quasi-compact) Let $Z = \overline{\varphi(X)} \subseteq Y$. $\mathscr{I}_Z = \ker(\varphi^\sharp : \mathscr{O}_Y \to \varphi_*\mathscr{O}_X)$. $\mathscr{I}_Z$ is quasicoherent implies that $(Z, \mathscr{O}_X/\mathscr{I}_Z) \subseteq Y$ is a closed subscheme called the scheme-theoretic image of X.

Exercise: X is reduced implies that $Z = \overline{\varphi(X)}$ is reduced.

Example: $Y = \operatorname{Spec} k[x,y]/(xy, y^2)$, $X = D(x) = \operatorname{Spec} k[x, x^{-1}] \subseteq Y$. $j : X \subseteq Y$. So $\overline{j(X)} = Y_{red} = \mathbb{A}^1$.

Application: $X \xrightarrow{f} Y \xrightarrow{g} Z$ an immersion. Then $X \subseteq \overline{gf(X)} \to Z$ is an immersion.

Exercise: Check This! Hint: Assume f is closed and g is open.

Lemma 6.52. *X Nötherian Scheme, $U \subseteq X$ open, $\mathscr{F}$ a coherent $\mathscr{O}_U$-module. Then there exists a coherent $\mathscr{O}_X$-module $\mathscr{F}'$ such that $\mathscr{F}'|_U \simeq \mathscr{F}$.*

Note: $i : U \to X$, so $i_*\mathscr{F}$ is quasicoherent has $(i_*\mathscr{F})|_U \simeq \mathscr{F}$.

Proof. If not, let $U \subseteq X$ be a maximal open set such that the lemma is false, and WLOG, $\mathscr{F}$ is a counterexample.

Let $Y \subseteq X$ be open affine, $Y \not\subseteq U$. Then $j : U \cap Y \subseteq Y$. $\mathscr{G} = j_*(\mathscr{F}|_{U \cap V})$ is a quasicoherent $\mathscr{O}_Y$-module, and $\mathscr{G}|_{U \cap Y} = \mathscr{F}|_{U \cap Y}$. Let $M = \Gamma(Y, \mathscr{G}) = \Gamma(U \cap Y, \mathscr{F})$ be a module over $A = \Gamma(Y, \mathscr{O}_X)$.

$\mathscr{G} = \tilde{M}$. Now, $U \cap Y = Y_{f_1} \cup \ldots \cup Y_{f_n}$, $f_i \in A$. Then $\tilde{M}_{f_i} = \mathscr{G}|_{Y_{f_i}} = \mathscr{F}|_{Y_{f_i}}$ is coherent. Thus M_{f_i} is a finitely generated A_{f_i}-module. Choose $m_1, \ldots, m_N \in M$ generated M_{f_i} for all i. Set $M' = (m_1, \ldots, m_N) \subseteq M$. $\mathscr{G}' = \tilde{M}'$ is a coherent $\mathscr{O}_Y$-module. $\mathscr{G}' \subseteq \mathscr{G}$ and $\mathscr{G}'|_{U \cap Y} = \mathscr{G}|_{U \cap Y} = \mathscr{F}|_{U \cap Y}$.

Set $X' = U \cup Y$. Define an $\mathscr{O}_{X'}$-module $\mathscr{F}'$ by $\mathscr{F}'(V) = \{(a,b) | a \in \mathscr{F}(U \cap V), b \in \mathscr{G}'(Y \cap V)$ with $a|_{U \cap Y \cap V} = b|_{U \cap Y \cap V} \in \mathscr{F}(U \cap Y \cap V)\}$.

$\mathscr{F}'|_U = \mathscr{F}$, and $\mathscr{F}'|_Y = \mathscr{G}'$, so $\mathscr{F}'$ is a coherent $\mathscr{O}_{X'}$-module. The contradicts that $\mathscr{F}$ is a maximal counterexample. □

Theorem 6.53. *X is a scheme of finite type over a Nötherian ring A. $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. Then $\mathscr{L}$ is ample iff $\mathscr{L}^{\otimes m}$ is very ample relative to* $\operatorname{Spec} A$ *for some $m > 0$.*

Proof. Assume that $\mathscr{L}^{\otimes m}$ is very ample over A. Then there exists an immersion $X \subseteq \bar{X} \to \mathbb{P}^n_A$ where the first is open and the second is closed, such that $\mathscr{L}^{\otimes m} \simeq \mathscr{O}_X(1)$.

Let $\mathscr{F}$ be coherent $\mathscr{O}_X$-module. We know that $\mathscr{O}_{\bar{X}}(1)$ is ample, so the lemma implies that there exists a coherent $\mathscr{O}_{\bar{X}}$-module $\bar{\mathscr{F}}$ such that $\bar{\mathscr{F}}|_X = \mathscr{F}$. $\bar{\mathscr{F}} \otimes \mathscr{O}_{\bar{X}}(N)$ generated by global sections for all $N >> 0$, so $\mathscr{F} \otimes \mathscr{O}_X(N)$ is generated by global sections for $N >> 0$. Thus $\mathscr{O}_X(1) = \mathscr{L}^{\otimes m}$ is ample, so $\mathscr{L}$ is ample.

Now we assume that $\mathscr{L}$ is ample. Let $P \in X$. There exists an open affine neighborhood $P \in U \subseteq X$ such that $\mathscr{L}|_U \simeq \mathscr{O}_U$. $Y = (X \setminus U)_{red} \to X$ is a closed subscheme, $\mathscr{I}_Y \subseteq \mathscr{O}_X$ is coherent implies htat $\mathscr{I}_Y \otimes \mathscr{L}^{\otimes n}$ is gneerated by global sections for some n. So there exists $s \in \Gamma(X, \mathscr{I}_Y \otimes \mathscr{L}^{\otimes n})$ such that $s_P \notin \mathfrak{m}_P(\mathscr{I}_Y \otimes \mathscr{L}^{\otimes n})_P$.

So $\mathscr{I}_Y \subseteq \mathscr{O}_X$, so $\mathscr{I} \otimes \mathscr{L}^{\otimes n} \subseteq \mathscr{L}^{\otimes n}$, thus $s \in \Gamma(X, \mathscr{L}^{\otimes n})$.

$P \in X_s \subseteq U$, so $(\mathscr{I}_Y)_P = \mathscr{O}_{X,P}$, $(\mathscr{I}_Y)_Q \subseteq \mathfrak{m}_Q$ for all $Q \in Y$, so $\mathscr{L}|_Y \simeq \mathscr{O}_U$. $s|_U$ corresponds to $f \in \Gamma(U, \mathscr{O}_X)$.

$X_s = U_s = U_f$ is affine. Therefore for all $P \in X$, there exists $n > 0$ and $s \in \Gamma(X, \mathscr{L}^{\otimes n})$ such that $P \in X_s$ and X_s affine. As X is Nötherian, $X = X_{s_1} \cup \ldots \cup X_{s_k}$, where $s_1, \ldots, s_k \in \Gamma(X, \mathscr{L}^{\otimes n})$ and X_{s_i} are affine. $n = \prod n_i$, so we can replace s_i with $s_i^{n/n_i} \in \Gamma(\mathscr{L}^{\otimes n})$.

WLOG, $n_i = n$ for all i. We replace $\mathscr{L}$ with $\mathscr{L}^{\otimes n}$. So WLOG, there exist $s_1, \ldots, s_k \in \Gamma(X, \mathscr{L})$ such that X_{s_i} is affine and $X = X_{s_1} \cup \ldots \cup X_{s_k}$.

Set $B_i = \Gamma(X_{s_i}, \mathscr{O}_X)$. As X is of finite type over $\operatorname{Spec} A$, we know that B_i is a finitely gneerated A-algebra generated by $b_{i1}, \ldots, b_{iN} \in \Gamma(X_{s_i}, \mathscr{O}_X)$. So there exists $n > 0$ and $c_{ij} \in \Gamma(X < \mathscr{O}_X \otimes \mathscr{L}^{\otimes n})$ such that $c_{ij}|_{X_{s_i}} = b_{ij} s_i^n$.

Now, $\mathscr{L}^{\otimes n}$ is an invertible $\mathscr{O}_X$-module generated by the global setions $\{s_i^n, c_{ij}\}$.

We define a morphism over A, $\varphi : X \to \mathbb{P}_A^{k(N+1)-1} = \operatorname{Proj} A[x_{ij} : 1 \leq i \leq n, 0 \leq j \leq N]$. Then $\varphi^*\mathscr{O}(1) = \mathscr{L}^{\otimes n}$, $\varphi^*(x_{ij}) = c_{ij}$ and $\varphi^*(x_{i0}) = s_i^n$. Note that $X_{s_i} = \varphi^{-1}(D_+(x_{i0})) \to D_+(x_{i0})$ is a closed immersion, so $B_i \leftarrow \mathscr{O}(D_+(x_{i0}))$ is surjecitve, mapping x_{ij}/x_{i0} to $c_{ij}/s_i^n = b_{ij}$.

Thus, $\varphi : X \to \cup_{i=1}^k D_+(x_{i0}) \subseteq \mathbb{P}_A^{k(N+1)-1}$ is an immersion, so $\mathscr{L}^{\otimes n} \simeq \varphi^*\mathscr{O}(1)$ is very ample. □

Remark: Y a scheme, $X \subseteq Y$ an integral closed subscheme. $D \subseteq Y$ is an effective Cartier divisor. Assume that $X \not\subseteq D$, then $D|_X = X \cap D \subseteq X$ is an effective Cartier divisor, and $\mathscr{L}(D)|_X = i^*\mathscr{L}(D) = \mathscr{L}(D \cap X)$. So $\mathscr{I}_D|_X = \mathscr{I}_{D \cap X}$.

Example: $\mathbb{P}^2_k = \operatorname{Proj} k[x, y, z]$. $X = V(zy^2 - x^3 + xz^2) \subseteq \mathbb{P}^2$, and $P_0 = (0 : 1 : 0)$ corresponds to $(x, z) \subseteq k[x, y, z]$. $\mathscr{L}(P_0)$ is ample, but not very ample.

Claim: $\mathscr{O}_X(1) \simeq \mathscr{L}(3P_0)$. Let $L = V(z) \subseteq \mathbb{P}^2$, this is just the line at infinity. $C\ell(\mathbb{P}^2) = \mathbb{Z}[L]$ and $\mathscr{O}_{\mathbb{P}^2}(1) = \mathscr{L}([L])$. So $\mathscr{O}_X(1) = \mathscr{O}_{\mathbb{P}^2}(1)|_X = \mathscr{L}([L])|_X = \mathscr{L}(L \cap X)$.

Show that $L \cap X = 3P_0 \in \mathrm{Div}(X)$. Notice that $X = X_y \cup X_z$ $\mathscr{I}_{L\cap X} \subseteq \mathscr{O}_X$ is generated by z/y on X_y and 1 on X_z.

$\mathscr{O}_{X,P_0} = k[x/y, z/y]_{(x/y,z/y)}/(z/y - (x/y)^3 + (x/y)(z/y)^3)$ and $\mathfrak{m}_{P_0} = (x/y)$. Then $v_{P_0}(L \cap X) = v_{P_0}(z/y) = 3$ as $z/y = (x/y)^3$ times a unit, therefore $L \cap X = 3P_0 \in \mathrm{Div}(X)$ so $\mathscr{L}(P_0)$ is ample.

Claim: $\mathscr{L}(P_0)$ is not very ample. Otherise there would exist $s \in \Gamma(X, \mathscr{L}(P_0))$ such that $s_{P_0} \notin \mathfrak{m}_{P_0}\mathscr{L}(P_0)_{P_0}$. $(s)_0 \subseteq X$ is an effective cartier divisor. If $\mathscr{L}(P_0)|_U = \mathscr{O}_U$ then $s|_U$ corresponds to $f \in \Gamma(U, \mathscr{O}_U)$, and $(s)_0 \cap U = V(f) \subseteq U$ is a closed subscheme. $(s)_0 \sim P_0$ so $\deg((s)_0) = \deg P_0 = 1$, so $(s_0) = Q \in \mathrm{Div}(X)$, $Q \neq P_0$. So $P_0 \sim Q$ and $P_0 \neq Q$ on a nonsingular projective curve X, so X is rational and $X \simeq \mathbb{P}^1$, contradiction.

Let $f : Y \to X$ be an affine morphism, $\mathscr{A} = f_*\mathscr{O}_Y$ is a quasi-coherent sheaf of $\mathscr{O}_X$ algebras. $f^\sharp : \mathscr{O}_X \to f_*\mathscr{O}_Y = \mathscr{A}$.

If $U \subseteq X$ open affine, then $V = f^{-1}(U) \subseteq Y$ is open affine, and $\mathscr{A}(U) = \mathscr{O}_Y(f^{-1}(U)) = \mathscr{O}_Y(V)$, thus, $f : V \to U$ given by $\mathscr{O}_X(U) \to \mathscr{A}(U)$.

Let X be any scheme, $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X$-algebra. Want: affine $f : Y \to X$ such that $\mathscr{A} = f_*\mathscr{O}_Y$.

Functor of Points

Let X be a scheme.

Definition 6.48 (Functor of Points)**.** *We define the functor of points to be a contravariant functor F_X :schemes to sets with $F_X(Y) = \hom_{sch}(Y, X)$. If $h : Y \to Y'$ is a morphism, then $F(h) = h^* : F_X(Y') \to F_X(Y)$ by $g \mapsto gh$.*

Example: X a variety over $k = \bar{k}$. Then $F_{X/k}(Y) = \hom_k(Y, X)$, then $F_{X/k}(\operatorname{Spec} k) = \{$the set of points of $X\}$.

Proposition 6.54. *The set of morphisms $\varphi : X \to X'$ are in correspondence with the natural transformations $\alpha : F_X \to F_{X'}$.*

Proof. Given $\varphi : X \to X'$, then $\alpha_Y : F_X(Y) \to F_{X'}(Y)$ can be defined by $g \mapsto \varphi g$.

Given $\alpha : F_X \to F_{X'}$, $\alpha_X : F_X(X) \to F_{X'}(X)$ set $\varphi = \alpha_X(\mathrm{id}) : X \to X'$.

Let Y be a scheme, and $g \in F_X(Y)$. As we have a natural transformation, we know that $\alpha_Y g^* = g^*\alpha_X$. Thus, if we take $\mathrm{id} \in F_X(X)$, it is mapped to $g \in F_X(Y)$ then to $\alpha_Y(g)$ in $F_X(Y)$. But also it is mapped to φ in $F_{X'}(X)$ and then to φg in $F_X(Y)$, so they must be equal, and so $\alpha_Y(g) = \varphi g$. □

Corollary 6.55. $X \simeq X' \iff F_X \simeq F_{X'}$.

Example: Let $X \times_S Y$ be the fibered product over $f : X \to S$ and $g : Y \to S$. Then $X \times_S Y$ is uniquely determined by $F_{X\times_S Y}(Z) = \{(p, q) | p : Z \to X, q : Z \to Y, fp = gq\} = F_X(Z) \times_{F_S(Z)} F_Y(Z)$.

X is a scheme, $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X$-algebra, $\gamma : \mathscr{O}_X \to \mathscr{A}$.

We define $F_{\mathscr{A}}$:schemes→sets by $F_{\mathscr{A}}(Y) = \{(f,\tilde{f}) | f : Y \to X, \tilde{F} : \mathscr{A} \to f_*\mathscr{O}_Y$ an $\mathscr{O}_X$-alg hom, and $f^\sharp : \mathscr{O}_X \xrightarrow{\gamma} \mathscr{A} \xrightarrow{\tilde{f}} f_*\mathscr{O}_Y\}$ on objects and if $h : Y \to Y'$ is a morphism, then we define $h^* : F_{\mathscr{A}}(Y') \to F_{\mathscr{A}}(Y)$ by $(f,\tilde{f}) \mapsto (fg, f_*(h^\sharp)\circ\tilde{f})$.

Definition 6.49 (Spec$\mathscr{A}$)**.** Spec$\mathscr{A}$ *is the unique scheme represented by* $F_{\mathscr{A}}$ *if it exists.* $F_{\mathrm{Spec}\mathscr{A}} = F_{\mathscr{A}}$.

Note: $F_{\mathrm{Spec}\mathscr{A}}(\mathrm{Spec}\mathscr{A}) = F_{\mathscr{A}}(\mathrm{Spec}\mathscr{A})$ so id maps to $(\pi,\tilde{\pi})$ the natural projection $\pi : \mathrm{Spec}\mathscr{A} \to X$, $\pi^\sharp : \mathscr{O}_X \xrightarrow{\gamma} \mathscr{A} \xrightarrow{\tilde{\pi}} \pi_*\mathscr{O}$ with $\mathscr{O} = \mathscr{O}_{\mathrm{Spec}\mathscr{A}}$.

Example: $f : Y \to X$ an affine morphism, $\mathscr{A} = f_*\mathscr{O}_Y$, $\gamma = f^\sharp : \mathscr{O}_X \to \mathscr{A}$, then Spec$\mathscr{A} = Y$

Let Z be any scheme, $g : Z \to Y$ a morphism, $g^\sharp : \mathscr{O}_Y \to g_*\mathscr{O}_Z$, so $f_*(g^\sharp) : f_*\mathscr{O}_Y = \mathscr{A} \to (gf)_*\mathscr{O}_Z$ define $F_Y(Z) \to F_{\mathscr{A}}(Z)$ by $g \mapsto (gf : Z \to X, f_*(g^\sharp))$.

Assume $(\varphi,\tilde{\varphi}) \in F_{\mathscr{A}}(Z)$. $\varphi : Z \to X$ and $\varphi^\sharp : \mathscr{O}_X \xrightarrow{f^\sharp} \mathscr{A} \xrightarrow{\tilde{\varphi}} \varphi_*\mathscr{O}_Z$.

Let $U \subseteq X$ be open affine, $V = f^{-1}(U) \subseteq Y$ affine.

$\tilde{\varphi} : \mathscr{A}(U) = \mathscr{O}_Y(V) \to \varphi_*\mathscr{O}_Z(U) = \mathscr{I}_Z(\varphi^{-1}(U))$ defines a morphism $\varphi^{-1}(U) \to V$.

Proposition 6.56. *Let* X *be a scheme,* $\mathscr{A}$ *be a quasi-coherent* $\mathscr{O}_X$*-algebra. Then* Spec$\mathscr{A}$ *exists,* $\pi : \mathrm{Spec}\mathscr{A} \to X$ *is affine, and* $\tilde{\pi} : \mathscr{A} \to \pi_*\mathscr{O}$ *is an isomorphism.*

Proof. If $U \subseteq X$ open affine, then $\mathrm{Spec}\mathscr{A}|_U = \mathrm{Spec}(\mathscr{A}(U))$, because $\pi : \mathrm{Spec}\,\mathscr{A}(U) \to X$ has $\pi_*\mathscr{O}_{\mathrm{Spec}\mathscr{A}} = \pi_*\tilde{\mathscr{A}(U)} = \tilde{\mathscr{A}(U)} = \mathscr{A}|_U$.

Assume $U' \subseteq U$ open subset, U affine, then $\mathrm{Spec}\mathscr{A}|_{U'} = \pi_U^{-1}(U')$ We have $\mathrm{Spec}\,\mathscr{A}|_{U'} = \mathrm{Spec}\,\mathscr{A}|_U \times_{U'} U \subseteq \mathrm{Spec}\,\mathscr{A}|_U \xrightarrow{\pi_U} U \supseteq U'$, so we glue $\{\mathrm{Spec}\,\mathscr{A}(U) | U \subseteq X \text{ affine }\}$ together to Spec$\,\mathscr{A}$. $U_1, U_2 \subseteq X$ open affine, and $\mathrm{Spec}\,\mathscr{A}(U_1) \supseteq \pi_{U_1}^{-1}(U_1 \cap U_2) = \mathrm{Spec}\,\mathscr{A}|_{U_1\cap U_2} = \pi_{U_2}^{-1}(U_1 \cap U_2) \subseteq \mathrm{Spec}\,\mathscr{A}(U_2)$. □

Let X be a Nötherian Scheme, $S = \oplus_{d\geq 0} S_d$ a graded quasi-coherent $\mathscr{O}_Y$-algebra. For $U \subseteq X$ open affine, $\pi_U : \mathrm{Proj}\,\mathscr{S}(U) \to U$. If $U' \subseteq U$ a smaller open affine, then $\mathscr{S}(U) \to \mathscr{S}(U')$ define $\mathrm{Proj}\,\mathscr{S}(U') \to \mathrm{Proj}\,\mathscr{S}(U)$. So we have a fiber square

$$\begin{array}{ccc} \mathrm{Proj}\,\mathscr{S}(U) & \xrightarrow{\pi_U} & U \\ \uparrow{\scriptstyle \mathrm{inc}} & & \uparrow{\scriptstyle \mathrm{inc}} \\ \mathrm{Proj}\,\mathscr{S}(U') & \xrightarrow{\pi_{U'}} & U' \end{array}$$

$\mathscr{S}$ is quasi coherent implies that $\mathscr{S}(U') = \mathscr{S}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{O}_X(U')$.

Define $F_{\mathscr{S}}(Y) = \hom_{sch}(Y, \mathrm{Proj}\,\mathscr{S}) = \{(f, g_U) | f : Y \to X, g_U : f^{-1}(U) \to \mathrm{Proj}\,\mathscr{S}(U) \text{ compatible}\}$.

Proposition 6.57. *There exists a unique scheme* $\mathrm{Proj}\,S$ *representing* F_S.

Proof. If $U \subseteq X$ open affine, then $\mathrm{Proj}(S|_U) = \mathrm{Proj}(S(U)) \xrightarrow{\pi_U} U$. If $U' \subseteq U$ any open subset, then $\mathrm{Proj}(S|_{U'}) = \pi_U^{-1}(U')$, now glue. □

Example: If X is a scheme, and $S = \mathscr{O}_X[\tau_0, \dots, \tau_n]$ then $\mathrm{Proj}(S) = \mathbb{P}^n_X = \mathbb{P}^n \times X$.

Example: $\mathbb{P}^1 = \mathrm{Proj}(k[x,y])$, $S = \oplus_{d\geq 0} S_d$. Let $a \in \mathbb{Z}$, then $S_d = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}(a) \oplus \mathscr{O}(2a) \oplus \dots \oplus \mathscr{O}(da)$. So if $f \in S_d, f' \in S_{d'}$, then $f \in \mathscr{O}(ia)$ and $f' \in \mathscr{O}(ja)$, so $ff' \in \mathscr{O}((i+j)a)$.

So we get $\pi : \mathrm{Proj}\, S \to \mathbb{P}^1$ with $\pi^{-1}(D_+(x)) = \mathrm{Proj}\, S(D_+(x)) = \mathrm{Proj}\, k[y/x][s, x^a t] = \mathbb{A}^1 \times \mathbb{P}^1$, and $\pi^{-1}(D_+(y)) = \mathrm{Proj}(k[x/y][s, y^a t]) = \mathbb{A}^1 \times \mathbb{P}^1$. Set $\lambda = y/x$, then this is $k[\lambda^{-1}][s, x^a \lambda^a t]$, so we glue along the $(\mathbb{A}^1 \setminus 0) \times \mathbb{P}^1$'s vis $(\lambda, (s:t)) \mapsto (\lambda^{-1}, (s : \lambda^a t))$.

Then $F_{-a} = \mathrm{Proj}(S) \xrightarrow{\pi} \mathbb{P}^1$ is the Hirzebruch Surface.

X is a scheme, $\mathscr{S} = \oplus_{d\geq 0} \mathscr{S}_d$ graded $\mathscr{O}_X$-algebra. Then $\underline{\mathrm{Proj}}\mathscr{S}$ is the unique scheme such that $\hom(Y, \underline{\mathrm{Proj}}\mathscr{S}) = F_{\mathscr{S}}(Y)$.

Note: Have $\mathscr{O}_U(1)$ on $\pi^{-1}(U)$, compatible: $U' \subseteq U$ a smaller open affine, then $\mathrm{Proj}\,\mathscr{S}(U') \subseteq \mathrm{Proj}\,\mathscr{S}(U)$, so $\mathscr{O}_{U'}(1)$ =pullback of $\mathscr{O}_U(1)$. Glue to get $\mathscr{O}(1)$ on $\underline{\mathrm{Proj}}\mathscr{S}$: $\Gamma(V, \mathscr{O}(1)) = \{(\sigma_U) | \sigma_U \in \Gamma(V \cap \pi^{-1}(U)), \mathscr{O}_U(1)$ which are compatible.

Remark: $S = \oplus_{d\geq 0} S_d$ is a graded ring, $u \in S_0$ a unit. Then define $\theta_d : S_d \to S_d$ by $\theta_d(s) = u^d s$. This gives an isomorphism of graded S_0-algebras $\theta : S \to S$.

Note: $h \in S_+$ homogeneous implies that $\theta = \mathrm{id} : S_{(h)} \to S_{(h)}$, so it induces $\mathrm{id} : \mathrm{Proj}\, S \to \mathrm{Proj}\, S$.

Definition 6.50. *Let $\mathscr{S} = \oplus_{d\geq 0} \mathscr{S}_d$ graded $\mathscr{O}_X$-algebra. $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. Then $\mathscr{S} * \mathscr{L} = \mathscr{S}' = \oplus_{d\geq 0} \mathscr{S}'_d$ where $\mathscr{S}'_d = \mathscr{S}_d \otimes \mathscr{L}^{\otimes d}$. Let $\pi : P = \underline{\mathrm{Proj}}\mathscr{S} \to X$ and $\pi' : P' = \underline{\mathrm{Proj}}\mathscr{S}' \to X$.*

Lemma 6.58. *We have an natural isomorphism $\varphi : P' \to P$ over X and $\mathscr{O}_{P'}(1) = \varphi^* \mathscr{O}_P(1) \otimes (\pi')^* \mathscr{L}$.*

Proof. Let $U \subset X$ open affine, with $\mathscr{O}_U \to \mathscr{L}|_U$ an isomorphism with 1 corresponding to $f \in \gamma(U, \mathscr{L})$. This defines an isomorphism $\mathscr{S}(U) \to \mathscr{S}'(U)$, with $\mathscr{S}(U)_d \to \mathscr{S}'(U)_d = \mathscr{S}(U)_d \otimes \Gamma(U, \mathscr{L}^{\otimes d})$ by $s \mapsto s \otimes f^d$.

This defines an isomorphism $\varphi : \mathrm{Proj}\,\mathscr{S}'(U) \to \mathrm{Proj}\,\mathscr{S}(U)$.

Remark implies that φ is independent of $f \in \Gamma(U, \mathscr{L})^*$. So we can glue to an isomorphism $\varphi : P' \to P$.

The sections of $\mathscr{O}_{P'}(1)$ correspond to elements in $\Gamma(U, \mathscr{S}'_1) = \Gamma(U, \mathscr{S}_1) \otimes \Gamma(U, \mathscr{L})$ correspond to sections of $\mathscr{O}_P(1) \otimes \pi^* \mathscr{L}$. ☐

Definition 6.51. *Assume X is Nötherian. $\mathscr{S} = \oplus_{d\geq 0} \mathscr{S}_d$ satisfies (+) if*

1. *$\mathscr{S}_0 = \mathscr{O}_X$*
2. *$\mathscr{S}_1$ is a coherent $\mathscr{O}_X$-module*
3. *$\mathscr{S}$ is generated (locally) by $\mathscr{S}_1$.*

Note: If $U \subseteq X$ is a small open affine, then $\exists \mathscr{S}_1^{\otimes d}|_U \to \mathscr{S}_d|_U \to 0$ implies that $\mathscr{S}_d$ is coherent.

Lemma 6.59. *Assume that $\mathscr{S}$ satisfies (+). Let $\pi : P = \operatorname{Proj} \mathscr{S} \to X$. Then π is prokper and if there exists an ample invertible $\mathscr{O}_X$-module $\mathscr{L}$ on X, then π is projective, and $\mathscr{O}_P(1) \otimes \mathscr{L}^{\otimes n}$ is very ample relative to X for some $n > 0$.*

Proof. Let $U \subset X$ open affine such that $\mathscr{S}(U)$ generated by $\mathscr{S}_1(U)$ as A-algebra, with $A = \mathscr{O}_X(U)$. Then $\mathscr{S}_1$-coherent implies that $\mathscr{S}_1(U)$ is a finitely gneerated A-module, so $\exists$ a graded A-algebra homomorphism $A[x_0, \ldots, x_N] \to \mathscr{S}(U)$ surjective. Thus, $\operatorname{Proj} \mathscr{S}(U) \overset{close}{\to} \mathbb{P}^N_U \overset{\pi}{\to} U$. Thus $\pi|_U$ is projective, so it is proper. Thus, $\pi : \operatorname{Proj} \mathscr{S} \to X$ is proper.

If $\mathscr{L}$ is ample, then $\mathscr{S}_1 \otimes \mathscr{L}^{\otimes n}$ is generated by global sections. As X is Nötherian, $\mathscr{S}_1 \otimes \mathscr{L}^{\otimes n}$ is coherent, so it is generated by finitely many global sections. Thus, there exists a surjection of graded $\mathscr{O}_X$-algberas $\mathscr{O}_X[T_0, \ldots, T_N] \to \mathscr{S} * (\mathscr{L}^{\otimes n})$, and so $P' = \operatorname{Proj}(S * \mathscr{L}^{\otimes n}) \to \operatorname{Proj} \mathscr{O}_X[T_i] = \mathbb{P}^n_X$ is closed, and so $\mathscr{O}_{P'}(1) = \varphi 6 * (\mathscr{O}_P(1) \otimes \pi^*(\mathscr{L}^{\otimes n}))$ is very ample. ☐

Definition 6.52 (Tensor Algebra)**.** *Let A be a ring and M be an A-module. Then $T^d M = M \otimes \ldots \otimes M$, d-times. $T(M) = \oplus_{d \geq 0} T^d M$ is called the tensor algebra. $S(M) = \oplus_{d \geq 0} S^d(M) = T(M)/(x \otimes y - y \otimes x)$ is the symmetric algbera. If $M = A^{\oplus r}$, then $S(M) = A[T_1, \ldots, T_r]$.*

If X is a Nötherian scheme, $\mathscr{E}$ a locally free $\mathscr{O}_X$-module of rank r and $\mathscr{E}^\vee = \mathscr{H}om(\mathscr{E}, \mathscr{O}_X)$ the dual sheaf is also locally free of rank r. Then $S(\mathscr{E}^\vee) = [U \mapsto S(\Gamma(U, \mathscr{E}^\vee))^+$ (sheafification) is a graded $\mathscr{O}_X$-algebra.

$S(\mathscr{E}^\vee)_0 = \mathscr{O}_X$, and $S(\mathscr{E}^\vee)_1 = \mathscr{E}^\vee$, ... this satisfies (+). Set $\pi : Y = \operatorname{Spec} S(\mathscr{E}^\vee) \to X$.

If $U = \operatorname{Spec} A \subseteq X$ open, $\mathscr{E}|_U \simeq \mathscr{O}_U^{\oplus r}$, then $S(\mathscr{E}^\vee)|_U \simeq \mathscr{O}_U[T_1, \ldots, T_r]$, so $\pi^{-1}(U) = \operatorname{Spec} \mathscr{O}_U[T_1, \ldots, T_r] = U \times \mathbb{A}^r$. Thus $\pi : Y \to X$ affine bundle (in fact, a vector bundle), so assume $f : U \to \pi^{-1}(U) \subseteq Y$ is a section, ($\pi f = \mathrm{id}_U$), then

$$\Gamma(U, \mathscr{E}^\vee) \to \Gamma(U, S(\mathscr{E}^\vee)) \overset{f^\sharp}{\to} \Gamma(U, \mathscr{O}_X)$$

gives an $\mathscr{O}_X$-homomorphism $f : \mathscr{E}^\vee \to \mathscr{O}_X$ over U, ie, a section $f \in \Gamma(U, \mathscr{E})$. THus $\mathscr{E}$ is the sheaf of sections of $\pi : \operatorname{Spec} S(\mathscr{E}^\vee) \to X$.

Definition 6.53. $\mathbb{P}(\mathscr{E}) = \underline{\operatorname{Proj}} S(\mathscr{E}^\vee)$.
$\pi : \mathbb{P}(\mathscr{E}) \to X$, $\mathscr{O}_{\mathscr{E}}(1)$.

If $\mathscr{E}|_U \simeq \mathscr{O}_U^{\otimes r}$, then $\pi^{-1}(U) = U \times \mathbb{P}^{r-1} = \mathbb{P}^{r-1}_U$.

Example: $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, $\pi : \mathbb{P}(\mathscr{L}) \to X$ an isomorphism, $\mathscr{O}_{\mathscr{L}}(1) = \pi^*(\mathscr{L}^{-1})$.

Proposition 6.60. *1. If $\operatorname{rank}(\mathscr{E}) \geq 2$, then $\pi_* \mathscr{O}_{\mathscr{E}} = S^m(\mathscr{E}^\vee)$ for $m \geq 0$ and 0 for $m < 0$.*

2. Have surjection $\pi^* \mathscr{E}^\vee \to \mathscr{O}_{\mathscr{E}}(1)$.

Proof. Have global $S^m(\mathscr{E}^\vee) \to \pi_* \mathscr{O}_{\mathscr{E}}(m)$, and in $U \subseteq X$ open affine, then $f \in \Gamma(U, S^m(\mathscr{E}^\vee))$ gives a section of $\mathscr{O}_{\mathscr{E}}(m)$ over $\pi^{-1}(U) = \operatorname{Proj} \Gamma(U, S(\mathscr{E}^\vee))$

isomorphism locally when $\mathscr{E}|_U \simeq \mathscr{O}_U^{\otimes r}$, $\pi^{-1}(U) = \mathbb{P}_U^{r-1}$, already computed $\Gamma(\mathbb{P}_U^{r-1}, \mathscr{O}(m))$.

For the secton part, we have an $\mathscr{O}_X$-homomorphism $\mathscr{E}^\vee \to \pi_*\mathscr{O}_{\mathscr{E}}(1)$ which gives an $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-homomorphism $\pi^*\mathscr{E}^\vee \to \mathscr{O}_{\mathscr{E}}(1)$, and can check that it is surjecitve locally. □

Universal Property

$\underline{F_{\mathbb{P}(\mathscr{E})}(Y)} = \{(f, \mathscr{L}, \theta) | f : Y \to X, \mathscr{L}$ an invertible $\mathscr{O}_Y$-module, and $\theta : f^*\mathscr{E}^\vee \to \mathscr{L}$ surjective$\}$. Left to reader.

Exercise: $\mathbb{P}_k^2, k = \bar{k}$. Then $\mathbb{P}^2 = \{L \subset k^3 | \dim L = 3\} = \{E_2 \subseteq k^3 | \dim E_2 = 2\}$.

$0 \to \mathscr{E} \to \mathscr{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathscr{O}_{\mathbb{P}^2}(1) \to 0$ with $\mathscr{E}$ locally free of rank 2, then $\mathbb{P}(\mathscr{E}) = F\ell(k^3) = \{(E_1, E_2) | E_1 \subset E_2 \subset k^3, \dim(E_i) = i\}$.

$\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^2$. Flag of locally free $\mathscr{O}_X$-module, $\mathscr{O}_{\mathscr{E}}(-1) \subseteq \pi^*\mathscr{E} \subseteq \mathscr{O}_{\mathbb{P}(\mathscr{E})}^{\oplus 3}$ correspond to flags of vector bundles $B_1 \subseteq B_2 \subseteq \mathbb{P}(\mathscr{E}) \times k^3$, $B_i = \{((E_1, E_2), \vec{v}) | \vec{v} \in E_i\}$.

7 Schemes II

Differentials

Let R be a ring, S a commutative R-algebra and M an S-module.

Definition 7.1 (R-derivation)**.** *$D : S \to M$ is an R-derivation if $D(fg) = fD(g) + gD(f)$, $D(f+g) = D(f) + D(g)$ for all $f, g \in S$ and $D(f) = 0$ if $f \in R$ (iff D is R-linear)*

Universal Derivation: Let F be the free S-module with basis $\{df | f \in S\}$, and $F' \subset F$ the submodule generated by $d(fg) - fdg - gdf, d(f+g) - df - dg$ and df for all $f \in R$. Then defime $\Omega_{S/R} = F/F'$ and $d = d_S = d_{S/R} : S \to \Omega_{S/R}$ by $d(F) = df + F'$. This satisfies the universal property that if $D : S \to M$ is any R-derivation, then $\exists! S$-homomorphism $\tilde{D} : \Omega_{S/R} \to M$ such that $D = \tilde{D} \circ d_{S/R}$.

Note: $P(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ and $f_1, \ldots, f_n \in S$. Then $D(P(f_1, \ldots, f_n)) = \sum_{i=1}^n \frac{\partial P}{\partial x_i}(f_1, \ldots, f_n) D(f_i)$.

Proposition 7.1. *$S = R[x_1, \ldots, x_n]$, then $\Omega_{S/R} \simeq S^{\oplus n} = \oplus_{i=1}^n S \cdot dx_i$ and $d : S \to S^{\oplus n}$ is given by $df = (df/dx_1, \ldots, df/dx_n)$.*

Proposition 7.2. *$R \to S \to T$ ring homomorphisms, then $\Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to \Omega_{T/S} \to 0$ is exact as T-modules.*

Proposition 7.3. *If S is an R-algebra and $T = S/I$, then $I/I^2 \xrightarrow{\delta} \Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to 0$ wotj δ taking $f + I^2$ to $df \otimes 1$.*

Proposition 7.4. *Let R' and S be R-algebras and $S' = S \otimes_R R'$. Then $\Omega_{S'/R'} = \Omega_{S/R} \otimes_S S'$*

Proposition 7.5. *$U \subseteq S$ a multiplicative subset, then $U^{-1}\Omega_{S/R} = \Omega_{U^{-1}S/R}$.*

Corollary 7.6. *If S is a localization of a finitely generated R-algebra, then $\Omega_{S/R}$ is a finitely generated S-module.*

Proof. $S = U^{-1}S'$, $S' = R[s_1, \dots, s_n]$. Then $\Omega_{S/R}$ is generated by $ds_1, \dots, ds_n$ as an S'-module, so $\Omega_{S/R} = U^{-1}\Omega_{S'/R}$ is generated by $ds_1, \dots, ds_n$ as an S-module. $\square$

Sheaves of Differentials

Let X be a topological space, $\mathscr{R}, \mathscr{S}$ sheaves of rings, and $\mathscr{R} \to \mathscr{S}$ a sheaf homomorphism.

Define $pre - \Omega = pre - \Omega_{\mathscr{S}/\mathscr{R}} = [U \mapsto \Omega_{S(U)/R(U)}]$. If $V \subseteq U$ is open, then we get a diagram $\mathscr{S}(U) \to \mathscr{S}(V) \to pre - \Omega(V)$, but also $\mathscr{S}(U) \to pre - \Omega(U)$, which goes to $pre - \Omega(V)$ by restriction, and this is an $\mathscr{R}(U)$-derivation. Then $pre - \Omega$ is a presheaf.

Define $\Omega_{\mathscr{S}/\mathscr{R}} = (pre - \Omega)^+$.

Let $\varphi : X \to Y$ be a morphism of schemes, $\varphi^\sharp : \varphi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$.

Define $\Omega_{X/Y} = \Omega_{\mathscr{O}_X/\varphi^{-1}\mathscr{O}_Y}$, and if we have $X \to \operatorname{Spec}(k)$ is a scheme over k, then $\Omega_X = \Omega_{X/k} = \Omega_{X/\operatorname{Spec}(k)}$. We call this the relative cotangent sheaf and the cotangent sheaf.

Proposition 7.7. *If $\varphi : X \to Y$ is a morphism of affine schemes, $X = \operatorname{Spec}(S)$ and $Y = \operatorname{Spec}(R)$, then $\Omega_{X/Y} \simeq \widetilde{\Omega_{S/R}}$.*

Corollary 7.8. *$\Omega_{X/Y}$ is always quasi-coherent. If $\varphi : X \to Y$ is locally of finite type, then $\Omega_{X/Y}$ is coherent.*

Proof. $P \in X$, take open affine neighborhoods $\varphi(P) \in V \subseteq Y$ adn $P \in U \subseteq \varphi^{-1}(V) \subseteq X$. $\Omega_{X/Y}|_U = \Omega_{U/V}$ is quasicoherent. If φ is lcoally of finite type, then we can take U, V such that $\mathscr{O}_X(U)$ finitely generated $\mathscr{O}_Y(V)$-algebra. $\square$

Proposition 7.9. *Take a fiber square:*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle g'}\uparrow & & \uparrow{\scriptstyle g} \\ X' & \xrightarrow{f} & Y' \end{array}$$

Then $g'^*\Omega_{X/Y} = \Omega_{X'/Y'}$

Proposition 7.10. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ morphisms then $f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$ is exact.*

Proposition 7.11. *If $g : Y \to Z$, $X \subseteq Y$ a closed subscheme, then $I_X/I_X^2 \to \Omega_{Y/Z} \otimes \mathscr{O}_X \to \Omega_{X/Z} \to 0$ is exact.*

Theorem 7.12. *Let Y be a scheme. $X = \mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times Y$. Then $\exists$ an exact sequence $0 \to \Omega_{X/Y} \to \mathscr{O}_X(-1)^{\oplus n+1} \to \mathscr{O}_X \to 0$.*

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \operatorname{Spec}\mathbb{Z} \end{array}$$

Proof. WLOG, $Y = \operatorname{Spec}(\mathbb{Z})$ and $X = \operatorname{Proj}(S)$ for $S = \mathbb{Z}[x_0, \ldots, x_n]$. Set $E = S(-1)^{\oplus n+1}$. We have a map $E \to S$ by $e_i \mapsto x_i$, it has kernel M, so we get an exact sequence $0 \to M \to E \to S$. This gives us an exact sequence $0 \to \tilde{M} \to \mathscr{O}(-1)^{\oplus n+1} \to \mathscr{O}_X \to 0$

Notice: $f \in \mathbb{Q}(x_0, \ldots, x_n)$ homogeneous of degree d, then $\sum_{i=0}^n \frac{\partial f}{\partial x_i} x_i = df$. Define $d : \mathscr{O}_X \to \tilde{E}$ by $d(f) = \sum_{i=0}^n \frac{\partial f}{\partial x_i} e_i$. Note that $d(\mathscr{O}_X) \subseteq \tilde{M}$, ie $d : \mathscr{O}_X \to \tilde{M}$ is a derivation. This induces $\tilde{d} : \Omega_{X/\mathbb{Z}} \to \tilde{M}$, we will check that this is an isomorphism locally on $D_+(x_i) = \operatorname{Spec}\mathbb{Z}[x_0/x_i, \ldots, x_n/x_i]$.

Enough to check that $\Gamma(D_+(x_i), \tilde{M})$ is a free $S_{(x_i)}$-module with basis $\{d(x_j/x_i) | j \neq i\}$.

$0 \to M_{(x_i)} \to E_{(x_i)} \to S_{(x_i)} \to 0$ which takes $e_j \to x_j$ but there's a map from $S_{(x_i)} \to E_{(x_i)}$ taking $1 \to e_i/x_i$. And therefore, $d(x_j/x_i) = \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i$. □

Singular Varieties

Let $k = \bar{k}$

Definition 7.2 (Nonsingular Variety)**.** *A variety X over k is nonsingular at $P \in X$ if $\mathscr{O}_{X,P}$ is a regular local ring. X is nonsingular if all points are nonsingular.*

Theorem 7.13. *X is an irreducible separated scheme of finite type over $k = \bar{k}$. Set $n = \dim(X)$. Then $\Omega_{X/k}$ is locally free of rank n iff X is a nonsingular variety.*

Recall that $F \subset K$ is a field extension, and each $a \in K$ has a minimal polynomial $p_a(T) \in F[T]$ such that $p_a(a) = 0 \in K$. K is separable over F if $p_a(T)$ does not have multiple roots for all $a \in K$.

Exercise: If K/F is separable, then $\Omega_{K/L} = 0$. ($0 = d_K(p_a(a)) = p_a'(a) d_K(a)$ with $p_a'(a) \neq 0$)

Corollary 7.14. *X is a variety over k implies that a dense open subset of X is nonsingular.*

Proof. $K = k(X)$.

FACT: k is a perfect field implies that any finitely generated extenseion $k \subset K$ is separaly generated. IE, there exists a transcendence basis $x_1, \ldots, x_n \in K$ suchj that $k(x_1, \ldots, x_n) \subseteq K$ is separable.

K is separable over $F = k(x_1, \ldots, x_n)$ for $n = \dim(X)$. Let $S = k[x_1, \ldots, x_n]$, $F = S_0 \Rightarrow \Omega_{F/k} = (\Omega_{S/k})_0 = F^{\oplus n}$.

So $\Omega_{F/k} \otimes_F K \xrightarrow{\alpha} \Omega_{K/k} \to \Omega_{K/F} = 0$.

As any k-derivation $D : F \to K$ can be extended to $D : K \to K$ we can conclude that α is injective.

Thus, $\Omega_{K/k} \simeq K^n$. $\mathrm{Spec}(R) \subseteq X$ open. $K = R_0 \Rightarrow (\Omega_{R/k})_0 = \Omega_{K/k} = K^n$. Thus $\exists 0 \neq f \in R : (\Omega_{R/k})_f = \Omega_{R_f/k} = (R_f)^n$. $\square$

Lemma 7.15. *If $(R, \mathfrak{m})$ is a local ring, $k = R/\mathfrak{m}$, and $k \subseteq R$. Then $\delta : \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\simeq} \Omega_{R/k} \otimes_R k$ is an isomorphism.*

Theorem 7.16. *X nonsing variety over $k = \bar{k}$ and $Y \subseteq X$ irreducible closed subscheme, then Y is nonsingular iff $\Omega_{Y/k}$ is locally free and $0 \to \mathscr{I}_Y/\mathscr{I}_Y^2 \to \Omega_X \otimes \mathscr{O}_Y \to \Omega_Y \to 0$ is exact.*

Proof. Assume the latter conditions, set $q = \mathrm{rank}(\Omega_Y)$, $n = \mathrm{rank}(\Omega_X) = \dim(X)$. It is enough to show that $q = \dim(Y)$. The second condition causes $\mathscr{I}_Y/\mathscr{I}_Y^2$ to be locally free of rank $n - q$, and by Nakayama, $\mathscr{I}_Y$ is locally generated by $n - q$ elements.

The Principle Ideal Theorem says that $\dim(Y) \geq q$. Let $P \in Y$, $\mathfrak{m}_P \subseteq \mathscr{O}_{Y,P}$, then the lemma says that $\mathfrak{m}_P/\mathfrak{m}_P^2 = \Omega_{Y,P} \otimes \mathscr{O}_{Y,P}/\mathfrak{m}_P = \Omega_{Y,P} \otimes k$. So $\dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = q$. Thus $\dim(Y) = \dim \mathscr{O}_{Y,P} \leq q$, so $\dim(Y) = q$, so Y is nonsingular.

Assume that Y is nonsingular. $\Omega_{Y/k}$ is then locally free of rank $q = \dim(Y)$. We know that $\mathscr{I}_Y/\mathscr{I}_Y^2 \to \Omega_X \otimes \mathscr{O}_Y \xrightarrow{\varphi} \Omega_Y \to 0$ is exact, so all that remains is showing that $\delta : \mathscr{I}_Y/\mathscr{I}_Y^2 \to \Omega_X \otimes \mathscr{O}_Y$ is injective.

Let $P \in Y$ be any closed point. Localize at P. Then $\ker \varphi_P$ is a free $\mathscr{O}_{Y,P}$-module of rank $r = n - q$, so there exist $x_1, \ldots, x_r \in \mathscr{I}_{Y,P}$ such that $dx_1, \ldots, dx_r$ form a basis for $\ker \varphi_P$. Let $I' \subseteq I_Y$ be a subideal generated by $x_1, \ldots, x_r$ and $Y' = Z(I') \subseteq X$.

Thus, $dx_1, \ldots, dx_r$ must generate a free subsheaf of rank r in $\Omega_X \otimes \mathscr{O}_{Y'}$. So we get that $I'/I'^2 \xrightarrow{\delta} \Omega_X \otimes \mathscr{O}_{Y'} \to \Omega_{Y'} \to 0$ and δ' must be injective.

So the first part tells us that Y' is a nonsingular variety and $\dim(Y') = \mathrm{rank}\, \Omega_{Y'} = n - r = q$, and $Y \subseteq Y' \subseteq X$ closed subschemes, Y, Y' both have dimension q, so $Y' = Y$ and $\delta' = \delta$. $\square$

Exercise: $(R, \mathfrak{m})$ is a regular local ring, $f \in \mathfrak{m}$. Then $R/(f)$ is regular local iff $f \notin \mathfrak{m}^2$. Use: Minimum number of gens of $\mathfrak{m} = \dim_k \mathfrak{m}/\mathfrak{m}^2$, $k = R/\mathfrak{m}$.

Theorem 7.17 (Bertini's Theorem)**.** *Let $X \subseteq \mathbb{P}_k^n$ a closed irreducible nonsingular subvariety, k algebraically closed. Then there exists a hyperplane $H \subseteq \mathbb{P}^n$ such that $X \not\subseteq H$ and $X \cap H$ is nonsingular.*

Proof. $V = \Gamma(\mathbb{P}^n, \mathscr{O}(1)) = kx_0 \oplus \ldots \oplus kx_n$. Then points of $\mathbb{P}(V)$ correspond to hyperplanes by $f \mapsto Z(f)$. Let $P \in X$ be a closed point, $B_P = \{f \in \mathbb{P}(V) | X \subseteq Z(f)$ or $P \in X \cap Z(f)$ is a singular point$\}$.

Will show: $\cup_{P \in X} B_P \subsetneq \mathbb{P}(V)$ is proper closed.

Set $r = \dim(X)$. Claim: $B_P \subseteq \mathbb{P}(V)$ is a linear subspace of dimension $n - r - 1$. Check that $f_0 \in V$ such that $P \notin Z(f_0) \subseteq \mathbb{P}^n$. For $f \in V$, we have

$f/f_0 \in \mathscr{O}_{X,P}$. Then $\mathscr{O}_{X\cap Z(f),P} = \mathscr{O}_{X,P}/(f/f_0)$. $f/f_0 \in \mathfrak{m}_P \iff P \in Z(f)$, $f/f_0 = 0 \in \mathscr{O}_{X,P} \iff X \subseteq Z(f)$, $f/f_0 \in \mathfrak{m}_P/\mathfrak{m}_P^2 \iff f \in B_P$.

Note: $\varphi_P : V \to \mathscr{O}_{X,P}/\mathfrak{m}_P^2$ by $f \mapsto f/f_0$ is surjective because $k = \bar{k}$ so $\mathfrak{m}_P$ is generated by linear forms.

$\dim_k V = n+1$, $\dim_k(\mathscr{O}_{X,P}/\mathfrak{m}_P^2) = r+1$. So $\dim \ker \varphi_P = n-r$, so $B_P = \mathbb{P}(\ker \varphi_P)$ so $\dim B_P = n-r-1$.

Now we define $B = \{(P,f) \in X \times \mathbb{P}(V) | f \in B_P\}$

Check: This is actually a closed subvariety of $X \times \mathbb{P}(V)$. $\pi_1 : B \to X$ is a surjective morphism and $\pi_1^{-1}(P) = B_P$ has $\dim B_P = n-r-1$ and $\dim(X) = r$ and so $\dim B = \dim(X) + \dim B_P = n-1$ and $\cup_{P\in X} B_P = \pi_2(B) \subseteq \mathbb{P}(V)$. Then $\pi_2 : X \times \mathbb{P}(V) \to \mathbb{P}(V)$ is a proper map, so $\pi_2(B)$ is closed of dimension at most $n-1$. ☐

8 Cohomology

Let R be a ring.

Definition 8.1 (Injective)**.** *An R-modules I is injective if for every R-module M and submodule M' and R-homomorphism $\varphi : M' \to I$, there exists an extension to M.*

Definition 8.2 (Divisible Abelian Group)**.** *A $\mathbb{Z}$-module T is divisible if for all $n \in \mathbb{Z}$ the map $T \to T$ by multiplication by n is surjective.*

Examples: $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}$.

Lemma 8.1. *Any divisible $\mathbb{Z}$-module is injective.*

Proof. $M' \subset M$, $\varphi' : M' \to T$. Consider $\{(N,\psi) | M' \subset N \subset M, \psi : N \to T$ extends $\varphi'\}$.

Zorn's Lemma says that there is a maximal element (N,ψ).

Claim: $N = M$. Else take $x \in M \setminus N$. $0 \to N \to (N,x) \to \mathbb{Z}/m\mathbb{Z} \to 0$. Either $m = 0$ in which case $(N,x) = N \oplus \mathbb{Z}x$ and we can extend $\varphi(x) = 0$ or $m \neq 0$ then $mn \in N$. So there exists $y \in T$ with $\varphi'(mx) = my$, so $\psi(x) = y$. ☐

Definition 8.3. *If M is a $\mathbb{Z}$-module, $M\hat{}= \hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.*

Exercise: $M \to M\hat{}\hat{}$, $x \mapsto [f \mapsto f(x)]$ is injective.

Corollary 8.2. *Every $\mathbb{Z}$-module M is contained in a divisible module.*

Proof. Take F free, $F \to M\hat{} \to 0$. So $M \to M\hat{}\hat{} \to F\hat{}$ =direct product of $\mathbb{Q}/\mathbb{Z}$s, with all the maps inclusions. ☐

Note: T a $\mathbb{Z}$-module, M an R-module, then $\hom_{\mathbb{Z}}(R,T)$ is an R-module.

$\hom_{\mathbb{Z}}(M,T) \simeq \hom_R(M, \hom_{\mathbb{Z}}(R,T))$ by $f \mapsto [x \mapsto [r \mapsto f(rx)]]$.

Lemma 8.3. *T a divisible $\mathbb{Z}$-module implies that $I = \hom_{\mathbb{Z}}(R,T)$ is an injective R-module.*

Proof. $M' \subseteq M$ are R-modules. Want:

$$\begin{array}{ccc} \hom_R(M, I) & \xrightarrow{\text{surj}} & \hom_R(M', I) \\ \uparrow \simeq & & \uparrow \simeq \\ \hom_{\mathbb{Z}}(M, T) & \xrightarrow{\text{surj}} & \hom_{\mathbb{Z}}(M', T) \end{array}$$

Which is true. □

Theorem 8.4. *Every R-module M is a submodule of an injective R-module.*

Proof. Lemma implies that there exists $f : M \to T$ injective $\mathbb{Z}$-hom when T is divisible. We define $M \to I = \hom_{\mathbb{Z}}(R, T)$ by $x \mapsto f_x$ with $f_x(r) = f(xr)$

Note that $M \subseteq I$: this is because f was injective to begin with. □

Definition 8.4 (Abelian Category)**.** *A category $\mathscr{C}$ is abelian if* $\hom(A, B)$ *is an abelian group for all $A, B \in obj(\mathscr{C})$,* $\hom(B, C) \times \hom(A, B) \to \hom(A, C)$ *is a group homomorphism, finite direct products exist, kernels exist, cokernels exist, etc. See Weibel.*

For any I an object of $\mathscr{C}$, we have a contravariant functor $\hom(-, I) : \mathscr{C} \to$ *Ab*. It is left exact, ie $0 \to A' \to A \to A'' \to 0$ exact implies $0 \to \hom(A'', I) \to \hom(A, I) \to \hom(A', I)$ exact.

Definition 8.5 (Enough Injectives)**.** *I is injective if* $\hom(-, I)$ *is exact.*

A category $\mathscr{C}$ has enough injectives if every object is a subobject of an injective object.

Corollary 8.5. *If $(X, \mathscr{O}_X)$ is a ringed space, then the category of $\mathscr{O}_X$-modules has enough injectives.*

Proof. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. For $p \in X$, there exists an inclusion $\mathscr{F}_P \to I_P$ an injective $\mathscr{O}_{X,P}$-module. Set $\mathfrak{I}(U) = \prod_{p \in U} I_P$.

Claim: $\mathfrak{I}$ is an injective $\mathscr{O}_X$-module. For any $\mathscr{O}_X$-module, $\mathscr{G}$, we have $\hom_{\mathscr{O}_X}(\mathscr{G}, \mathfrak{I}) = \prod_{p \in X} \hom_{\mathscr{O}_X} \hom_{\mathscr{O}_{X,P}}(\mathscr{G}_P, \mathfrak{I}_P)$.

So if $\mathscr{G}' \subseteq \mathscr{G}$ submodule, then $\hom(\mathscr{G}, \mathfrak{I}) \to \hom(\mathscr{G}', \mathscr{I})$ is surjective. □

Corollary 8.6. *Let X be a topological space, then the category $Ab(X)$ of sheaves of abelian groups on X has enough injectives.*

Proof. Set $\mathscr{O}_X$ to be the constant sheaf $\mathbb{Z}$, the sheaf of locally constant functions to $\mathbb{Z}$. □

Injective Resolution

Let M be an object of $\mathscr{C}$, then if $\mathscr{C}$ has enough injectives, there exists an exacct sequence $0 \to M \to I^0 \to I^1 \to \ldots$ with I^j injective, that is, M has an injective resolution.

Let $F : \mathscr{C} \to \mathscr{D}$ be a left exact functor. Then we have a complex $F(I^\cdot) = 0 \to F(I^0) \to F(I^1) \to \ldots$.

Definition 8.6. $R^jF(M) = H^j(F(I^\cdot))$.

Example: Left Exact implies that $0 \to F(M) \to F(I^0) \to \ldots$. We throw away M so that $H^0(I^\cdot) = R^0F(M) = F(M)$.

Definition 8.7. *Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups, and $\Gamma : Ab(X) \to Ab$ by $\Gamma(\mathscr{F}) = \Gamma(X, \mathscr{F})$ is left exact. Then set $H^j(X, \mathscr{F}) = R^j\Gamma(\mathscr{F})$.*

Let $A^\cdot, B^\cdot$ be complexes in $\mathscr{C}$, a morphism $f : A^\cdot \to B^\cdot$ is a collection of $f_n : A^n \to B^n$ such that the appropriate squares all commute.

This induces a map $H(f) : H^n(A^\cdot) \to H^n(B^\cdot)$.

$f, g : A^\cdot \to B^\cdot$ are homotopic ($f \sim g$) iff there exists morphisms $h_n : A^n \to B^{n-1}$ with $f_n - g_n = d_B^{n-1}h_n + h_{n+1}d_A^n$. This implies that $H(f) = H(g) : H^n(A^\cdot) \to H^n(B^\cdot)$.

Lemma 8.7. *Consider two chain complexes $A^\cdot$ and $I^\cdot$, $0 \to M \to A^\cdot$ and $0 \to M' \to I^\cdot$ an injective resolution, with $\varphi : M \to M'$. Then there exists a morphism $f : (M \to A^\cdot) \to (M' \to I^\cdot)$ and any other is homotopic to f.*

Proof. See Weibel. □

Note: $(f_1 - g_1 - d_B^0 h_1)d_A^0 = d_B^0(f_0 - g_0 - h_1 d_A^0) = d_B^0 d_B^{-1} h_0 = 0$.

Corollary 8.8. *$R^jF(M)$ is independent of the choice of injective resolution.*

Corollary 8.9. *A morphism $\varphi : M \to M'$ induces a unique $\varphi_* : R^jF(M) \to R^jF(M')$*

Lemma 8.10 (Horseshoe Lemma)**.** *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence in a category with enough injectives. Then there exists...*

Theorem 8.11. *Let $\mathscr{C}$ be abelian with enough injectives and $F : \mathscr{C} \to \mathscr{C}'$ a left exact functor.*

1. *$R^nF : \mathscr{C} \to \mathscr{C}'$ is additive, ie, $R^nF(M \oplus M') = R^nF(M) \oplus R^nF(M')$*
2. *$R^0F(M) = F(M)$ and $F^nF(I) = 0$ if I is injective and $n \geq 1$.*
3. *$0 \to M' \to M \to M'' \to 0$ short exact gives a long exact sequence $0 \to F(M') \to F(M) \to F(M'') \to R^1F(M') \to \ldots$*
4. *The δ morphisms from $R^nF(M'') \to R^{n+1}F(M')$ are natural.*

Definition 8.8 (F-acyclic)**.** *$J \in ob\mathscr{C}$ is F-acyclic if $R^nF(J) = 0$ for all $n \geq 1$.*

Example: J injective implies that J is F-acyclic for all left exact functors F.

Lemma 8.12. *$0 \to Y^0 \to Y^1 \to \ldots$ exact with Y^i F-acyclic implies that $0 \to F(Y^0) \to F(Y^1) \to \ldots$ is exact.*

Theorem 8.13. *$0 \to M \to J^0 \to \dots$ a resolution by F-acyclic objects J^n. Then $R^nF(M) = H^n(F(J^\cdot))$*

Definition 8.9 (δ-functor)**.** *If $\mathscr{C}$ and $\mathscr{C}'$ are abelian categories, then a δ-functor from $\mathscr{C} \to \mathscr{C}'$ is a sequence of functors $T = (T^n : \mathscr{C} \to \mathscr{C}')_{n\geq 0}$ together with a morphism $\delta^n : T^n(M'') \to T^{n+1}(M')$ for each short exactr sequence $0 \to M' \to M \to M'' \to 0$ such that*

1. *Long exact sequence*
2. *Naturality.*

Example: $T^n = R^nF$.

Definition 8.10 (Universal δ-functor)**.** *A δ-functor T is universal if, given any δ functor U and natural transformation $f_0 : T^0 \to U^0$, then there are unique $f_i : T^i \to U^i$ such that the appropriate diagram commutes.*

Definition 8.11 (Erasable)**.** *An additive functor $F : \mathscr{C} \to \mathscr{C}'$ is erasable if for all $M \in ob(\mathscr{C})$ there exists a monomorphism $P : M \to J$ such that $F(P) = 0 : F(M) \to F(J)$.*

Example: R^nF is erasable for $n \geq 1$, $M \to I$ a monomorphism and I injective, then $R^nF(M) \to R^nF(I) = 0$.

Theorem 8.14. *Let $T = (T^n)$ be a δ-functor. If every T^n is erasable for all $n \geq 1$ then T is a universal δ-functor.*

Proof. Assume that $U = (U^n)$ is a δ-functor with $f_0 : T_0 \to U_0$ a natural transformation. Let $M \in ob(\mathscr{C})$.

$0 \to M \xrightarrow{p} J \xrightarrow{q} X \to 0$ such that $T^1(p) = 0$.

Then we take the long exact sequences, horizontally, and f_0 vertically. We use that $\ker \delta^0_T = \operatorname{Im}(q_*) \subseteq \ker \delta^0_U \circ f_0$. Then $\exists! f_1(M) : T^1(M) \to U^1(M)$. Must check that f_1 is a natural transformation.

Check: f_1 commutes with δ.

To get f_2, we use $0 \to M \to J \to X \to 0$ such that $T^2(p) = 0$. Proceed by induction using the long exact sequence. □

Definition 8.12. *Let X be a topological space, and $\mathscr{F}$ a sheaf of abelian groups (hereafter referred to as an abelian sheaf). Let $\Gamma = \Gamma(X, -)$. Then $H^n(X, \mathscr{F})$ is defined to be $R^n\Gamma(\mathscr{F})$.*

Definition 8.13 (Flasque)**.** *A sheaf $\mathscr{F}$ is flasque if all the restriction maps are surjective.*

Lemma 8.15. *$(X, \mathscr{O}_X)$ a ringed space. $\mathscr{F}$ injective as an $\mathscr{O}_X$-module. Then $\mathscr{F}$ is flasque.*

Proof. For $U \subseteq X$ open, Set $\mathscr{O}_U = j_!(\mathscr{O}_X|_U)$ by $j : U \to X$ the inclusion and for $W \subseteq X$ open, $\mathscr{O}_U(W) = \mathscr{O}_X(W)$ if $W \subseteq U$ and 0 else.

Note, $\hom(\mathscr{O}_U, \mathscr{F}) = \mathscr{F}(U)$. If $V \subseteq U$, then $0 \to \mathscr{O}_V \to \mathscr{O}_U$ implies that $\hom(\mathscr{O}_U, \mathscr{F}) \to \hom(\mathscr{O}_V, \mathscr{F})$ is surjective. □

Lemma 8.16. *Assume that* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *exact.*

1. $\mathscr{F}'$ *is flaque implies that* $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \to 0$ *is exact*
2. $\mathscr{F}'$ *and* $\mathscr{F}$ *flasque implies that* $\mathscr{F}''$ *is flasque.*

Proof. Let $\gamma \in \Gamma(X, \mathscr{F}')$. Consider $\{(U, \beta) | U \subseteq X$ open and $\beta \in \Gamma(U, \mathscr{F})$ with $\beta \mapsto \gamma|_U \in \mathscr{F}''(U)\}$.

Then $(U', \beta') \leq (U, \beta)$ iff $U' \subseteq U$ and $\beta' = \beta|_{U'}$. Zorn's lemma gives us a maximal element (U, β).

Claim: $U = X$. Assume not, then there exists (V, σ) such that $V \not\subseteq U$ and $\sigma \in \Gamma(V, \mathscr{F})$, $\sigma \mapsto \gamma|_V$. Then $\beta|_{U\cap V} - \sigma|_{U \cap V} \in \mathscr{F}'(U \cap V)$. Then flasque implies that $\exists \alpha \in \mathscr{F}'(V)$ with $\alpha|_{U\cap V} = \beta - \sigma$. Set $\beta_1 = \sigma + \alpha \in \mathscr{F}(V)$. Then $\beta|_{U\cap V} = \beta_1|_{U \cap V}$, so glue β, β_1 to $\tilde{\beta} \in \Gamma(U \cup V, \mathscr{F})$, then $(U \cup V, \tilde{\beta}) \geq (U, \beta)$, contradiction. □

Proposition 8.17. *If* $\mathscr{F}$ *is flasque, then* $H^n(X, \mathscr{F}) = 0$ *for all* $n \geq 1$.

Proof. $0 \to \mathscr{F} \to I \to \mathscr{G} \to 0$ is a short exact sequence, with I injective and $\mathscr{G}$ flasque. So we get a long exact sequence $0 \to \mathscr{F}(X) \to I(X) \to \mathscr{G}(X) \to H^1(X, \mathscr{F}) \to \dots$. But $H^i(X, I) = 0$, so we get $H^1(X, \mathscr{F}) = 0$, $H^2(X, \mathscr{F}) = H^1(X, \mathscr{G}) = 0$, etc. □

Corollary 8.18. *If* $0 \to \mathscr{M} \to \mathscr{F}^0 \to \dots$ *is a flasque resolution, then* $H^n(X, \mathscr{M}) = H^n(\Gamma(X, \mathscr{F}^*))$.

Proposition 8.19. $(X, \mathscr{O}_X)$ *a ringed space. The right derived functos fo* $\Gamma : Mod(X) \to Mod(\mathscr{O}_X(X))$ *are given as* $\mathscr{M} \mapsto H^n(X, \mathscr{M})$.

Proof. Injective resolution in $Mod(X)$ $0 \to \mathscr{M} \to I^0 \to \dots$, we know that I^n is flasque for all n, so the corollary implies this. □

Definition 8.14 (Direct Limit Sheaf)**.** $\varinjlim \mathscr{F}_\alpha = [U \mapsto \varinjlim \mathscr{F}_\alpha(U)]^+$.

Lemma 8.20. X *Nötherian implies that* $\Gamma(U, \varinjlim \mathscr{F}_\alpha) = \varinjlim \Gamma(U, \mathscr{F}_\alpha)$.

Proof. Check sheaf axioms using finite open covers. □

Lemma 8.21. X *Nötherian and* $\mathscr{F}_\alpha$ *flasque, then* $\varinjlim \mathscr{F}_\alpha$ *is flasque.*

Proof. $V \subseteq U \subseteq X$ open, $\mathscr{F}_\alpha(U) \to \mathscr{F}_\alpha(V)$ surjective implies that $\varinjlim \mathscr{F}_\alpha(U) \to \varinjlim \mathscr{F}_\alpha(V)$ is surjective. □

Proposition 8.22. X *Nötherian,* $\{\mathscr{F}_\alpha\}$ *directed system, then* $H^n(X, \varinjlim \mathscr{F}_\alpha) = \varinjlim H^n(X, \mathscr{F}_\alpha)$.

Proof. Let $I = ind_A(Ab(X))$ be the category of all directed systems of abelian sheaves.

$T^n : I \to Ab$ by $T^n(\{\mathscr{F}_\alpha\}) = \varinjlim H^n(X, \mathscr{F}_\alpha)$ and $U^n : I \to Ab$ by $U^n(\{\mathscr{F}_\alpha\}) = H^n(X, \varinjlim \mathscr{F}_\alpha)$.

Note that $\varinjlim$ is exact, so T^n and U^n form δ-functors.

And $T^0(\{\mathscr{F}_\alpha\}) = \varinjlim \Gamma(X, \mathscr{F}_\alpha) = \Gamma(X, \varinjlim \mathscr{F}_\alpha) = U^0(\{\mathscr{F}_\alpha\})$.

By theorem from before, ETS that T^n and U^n are universal δ-functors, because they agree on the 0th term.

Given $\mathscr{F} \in Ab(X)$, set $\tilde{\mathscr{F}}(U) = \prod_{p\in U} \mathscr{F}_p$, the sheaf of discontinuous sections.

$0 \to \mathscr{F} \to \tilde{\mathscr{F}}$ exact and $\tilde{\mathscr{F}}$ is flasque.

$\{\mathscr{F}_\alpha\} \in ob(I)$, and $\{\tilde{\mathscr{F}}_\alpha\} \in ob(I)$, and we have $0 \to \{\mathscr{F}_\alpha\} \to \{\tilde{\mathscr{F}}_\alpha\}$.

So now sett that $T^n(\{\tilde{\mathscr{F}}_\alpha\}) = \varinjlim H^n(X, \tilde{\mathscr{F}}_\alpha) = 0$ and simialry for U^n. □

Lemma 8.23. *$Y \subseteq X$ a closed subspace, $j : Y \to X$ the inclusion, and $\mathscr{F}$ an abelian sheaf on Y. Then $H^n(Y, \mathscr{F}) = H^n(X, j_*\mathscr{F})$.*

Proof. $0 \to \mathscr{F} \to I^0 \to I^1 \to \ldots$ an injective resolution on Y. I^n is flasque, so $j_*(I^n)$ is flasque. So $0 \to j_*\mathscr{F} \to j_*I^0 \to \ldots$ is a flasque resolution (exactness follows from Y being closed).

So $H^n(Y, \mathscr{F}) = H^n(\Gamma(Y, I^*)) = H^n(\Gamma(X, j_*I^*)) = H^n(X, j_*\mathscr{F})$ □

Remark: $Y \subseteq X$ closed, $U = X \setminus Y$. Then $\mathscr{F}$ an abelian sheaf on X, we have $j : Y \to X$ and $i : U \to X$ inclusions. Then we can set $\mathscr{F}_U = i_!(\mathscr{F}|_U) = 0$ unless $V \subseteq U$, and if $V \subseteq U$, then $\mathscr{F}(V)$ and also $\mathscr{F}_Y = j_*(\mathscr{F}|_Y)$ where $\mathscr{F}_Y(U) = 0$ if $V \subseteq U$ and equals $j^{-1}\mathscr{F}(Y \cap V)$ if $V \not\subseteq U$. Then $\mathscr{F}_{Y,P} = \mathscr{F}_P$ if $P \in Y$ and zero else, as $\mathscr{F}_U = \mathscr{F}_p$ iff $p \in U$ and is 0 otherwise.

So we have $0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0$ is exact.

Theorem 8.24. *Let X be a Nötherian topological space, and $\mathscr{F}$ an abelian sheaf on X. Then $H^n(X, \mathscr{F}) = 0$ for all $n > \dim(X)$.*

Proof. Step 1: Reduce to X irreducible: If X is reducible, let Y be a maximal component in X. Take $U = X \setminus Y$, then we have the above short exact sequence. This gives us a long exact sequence on cohomology, $H^n(X, \mathscr{F}_U) \to H^n(X, \mathscr{F}) \to H^n(X, \mathscr{F}_Y) = H^n(Y, \mathscr{F}|_Y)$. If the theorem is true for irreducible spaces, then the last one is zero, and so we have isomorphisms $H^n(X, \mathscr{F}_U) \to H^n(X, \mathscr{F})$, so by induction, it is enough to show that the theorem holds when X is irreducible.

Step 2: $\dim X = 0$, X irreducible implies that X has open sets $\emptyset, X$ and nothing else. Then $Ab(X) = Ab$, so $\Gamma : Ab(X) \to Ab$ is the identity, and so it is exact, so $H^n(X, \mathscr{F}) = 0$ for $n \geq 1$.

Step 3: Assume that X is irreducible of dimension > 0. If $\sigma \in \mathscr{F}(U)$, then get $\mathbb{Z}_U \to \mathscr{F}$ by $1 \mapsto \sigma$. Let $\alpha = \{\sigma_1, \ldots, \sigma_m\}$ with $\sigma_i \in \mathscr{F}(U_i)$. Define $\mathscr{F}_\alpha = \mathrm{Im}(\oplus \mathbb{Z}_{U_i} \to \mathscr{F})$ the subsheaf generated by α. Set $B = \coprod_{U \subseteq X} \mathscr{F}(U)$. $A = \{\alpha \subseteq B | \alpha \text{ finite}\}$. A is a directed poset, so $\{\mathscr{F}_\alpha\}$ is a directed system. $\varinjlim \mathscr{F}_\alpha = \mathscr{F}$. And ten, as $H^n(X, \mathscr{F}) = H^n(X, \varinjlim \mathscr{F}_\alpha) = \varinjlim H^n(X, \mathscr{F}_\alpha)$, it suffices to prove this for $\mathscr{F} = \mathscr{F}_\alpha$ finitely generated.

If $\alpha' \subseteq \alpha$, then we have $0 \to \mathscr{F}_{\alpha'} \to \mathscr{F}_\alpha \to \mathscr{G} \to 0$, with $\mathscr{G}$ generated by the number of sections in $\alpha \setminus \alpha'$. So WLOG, $\mathscr{F}$ is generated by one section. So we have $0 \to \mathscr{K} \to \mathbb{Z}_U \to \mathscr{F} \to 0$ exact. Thus, it is enough to show vanishing for $\mathscr{F} \subseteq \mathbb{Z}_U$ a subsheaf.

For $p \in U$, $\mathscr{F}_p \subseteq \mathbb{Z}_{U,p} = \mathbb{Z}$. Define $d = \min\{e \in \mathbb{Z}_+ | \mathscr{F}_p = e\mathbb{Z} \text{ for some } p\}$. Choose p such that $\mathscr{F}_p = d\mathbb{Z}$, $d = \sigma_p$, $\sigma \in \mathscr{F}(V) \subseteq \mathbb{Z}_U(V)$, $p \in V$. Must have $\mathscr{F}|_V = d\mathbb{Z}_U|_V = d\mathbb{Z}_V$.

So we have $0 \to \mathbb{Z}_V \xrightarrow{\cdot d} \mathscr{F} \to \mathscr{F}/\mathbb{Z}_V \to 0$. $\mathrm{Supp}(\mathscr{F}/\mathbb{Z}_V) \subseteq \overline{U \setminus V}$, so $\dim \overline{U \setminus V} < \dim(X)$. So induction on dimension gives us that $H^n(X, \mathscr{F}/\mathbb{Z}_V) = 0$ for $n > \dim(X)$.

Set $Y = X \setminus U$, $0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_Y \to 0$ is ses, $\dim Y < \dim X$, so $H^n(X, \mathbb{Z}_Y) = 0$ for $n \geq \dim(X)$. $H^{n-1}(\mathbb{Z}_Y) \to H^n(\mathbb{Z}_U) \to H^n(\mathbb{Z}_X)$. So WLOG, $\mathscr{F} = \mathbb{Z}_X$.

If X is irreducible, then $\mathbb{Z}_X$ is flasque. So $H^n(X, \mathbb{Z}_X) = 0$ for all $n \geq 1$. □

NOTE: Above, $\mathscr{F}_U$ is not a sheaf, to correct things, sheafify

Lemma 8.25. *$Z \subseteq X$ closed, $\mathscr{G}$ a sheaf on Z, then $H^n(Z, \mathscr{G}) = H^n(X, j_*\mathscr{G})$.*

Note: $\mathrm{Supp}(\mathscr{F}_U) = U \subseteq \bar{U}$, then $\mathscr{F}_U = j_*(\mathscr{F}_U|_{\bar{U}})$, so $H^n(X, \mathscr{F}_U) = H^n(\bar{U}, \mathscr{F}_U|_{\bar{U}})$.

Next: If $X = \mathrm{Spec}(A)$ and $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, then $H^n(X, \mathscr{F}) = 0$ for $n \geq 1$.

Exercise: If M is an A-module, assume $\hom_A(A, M) \to \hom_A(I, M)$ is surjective for all ideals $I \subseteq A$. Then M is injective. Hint: $N' \subseteq N$, $\varphi : N' \to M$.

Let $M' \subseteq M$ an A-module, $I \subseteq A$ an ideal, $I^p(M' \cap I^n M) \subseteq M' \cap I^{n+p} M$ for all $p, n \geq 0$. The Artin-Rees lemma says that if A is Nötherian and M finitey generated, then for some $n > 0$ we have equality for all $p \geq 0$.

Definition 8.15. $\Gamma_I(M) = \{m \in M | I^n m = 0$ *for some* $n > 0\}$.

Lemma 8.26. *A Nötherian, M injective implies that $\Gamma_I(M)$ is injective.*

Proof. Let $J \subseteq A$ be an ideal. $\varphi : J \to \Gamma_I(M)$ an A-homomorphism. If $a \in J$, then $I^n \varphi(a) = 0$.

J finitely generated implies that $\exists p > 0$ such that $\varphi(I^p J) = 0$. By Artin-Rees, there exists $n > 0$ such that $I^p(J \cap I^n A) = J \cap I^{p+n} A$.

So we have $\varphi(J \cap I^{n+p}) = 0$. So the map from $J \to A$ gives a map from $J/J \cap I^{n+p} \to A/I^{n+p}$. As we have $J/J \cap I^{n+p} \to \Gamma_I(M) \subseteq M$, and M injective, we get a map $A/I^{n+p} \to M$. The image is contained in $\Gamma_I(M)$, as $I^{n+p}\psi(A/I^{n+p}) = 0$, and so the inclusion really extends to a map to $\Gamma_I(M)$. □

Lemma 8.27. *M an injective A-module, A Nötherian, $f \in A$. Then $\theta : M \to M_f$ is surjective.*

Proof. $\mathrm{Ann}(f) \subseteq \mathrm{Ann}(f^2) \subseteq \ldots \subseteq \mathrm{Ann}(f^r) = \mathrm{Ann}(f^{r+1}) = \ldots$. Let $x \in M_f$, then $x = \theta(y)/f^n$ for some $y \in M$. So we have a ses $0 \to \mathrm{Ann}(f^r) \to A \to (f^{n+r}) \to 0$. Then $A \to M$ by $f^r y$ and $(f^{n+r}) \subseteq A$, so we get a map $(f^{n+r}) \to M$ by $f^{n+r} \mapsto f^r y$. This map can be extended to A, call the image of 1 in this map z. Then $f^{n+r} z = f^r y$, then $\theta(z) = \theta(y)/f^n = x$. □

$\mathscr{F}$ a sheaf on X, $\sigma \in \mathscr{F}(U)$. $\mathrm{Supp}(\sigma) = \{p \in U | \sigma_p \neq 0 \in \mathscr{F}_p\} \subseteq U$ relatively closed. For $Z \subseteq X$ closed, $\Gamma_Z(U, \mathscr{F}) = \{\sigma \in \mathscr{F}(U) | \mathrm{Supp}(\sigma) \subseteq Z\}$

Definition 8.16. *Sheaf $\mathscr{H}^0_Z(\mathscr{F})$ by $\Gamma(U, \mathscr{H}^0_Z(\mathscr{F})) = \Gamma_Z(U, \mathscr{F})$.*

Let $j : X \setminus Z \to X$, then $0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_{X\setminus Z})$, which becomes short exact if $\mathscr{F}$ is flasque.

Assume $X = \operatorname{Spec}(A)$ and $\mathscr{F} = \tilde{M}$. For $m \in M = \Gamma(X, \tilde{M})$, then $\operatorname{Supp}(m) = V(\operatorname{Ann}(m))$, $p \in \operatorname{Supp}(m) \iff m/1 \neq 0 \in M_p \iff \operatorname{Ann}(m) \subseteq P$.

Lemma 8.28. *$X = \operatorname{Spec}(A)$, Nötherian, $Z = V(I) \subseteq X$. Then $\mathscr{H}^0_Z(\tilde{M}) = \widetilde{\Gamma_I(M)}$*

Proof. $0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_{U\setminus Z})$ tells us that $\mathscr{H}^0_Z$ is the kernel of a map of quasi-coherent sheaves.

Enough to show that $\Gamma(X, \mathscr{H}^0_Z(\tilde{M})) = \Gamma_I(M)$, $m \in \Gamma_Z(X, \tilde{M}) \iff \operatorname{Supp}(m) \subseteq Z$ iff $V(\operatorname{Ann}(m)) \subseteq V(I)$, iff $\sqrt{\operatorname{Ann}(m)} \supseteq \sqrt{I}$ iff $I^n \subseteq \operatorname{Ann}(m)$ (by Nötherian property), and this is iff $I^n m = 0$, iff $m \in \Gamma_I(M)$. □

Proposition 8.29. *A a Nötherian ring, $X = \operatorname{Spec}(A)$. Then M an injective A-module implies that $\tilde{M}$ is a flasque $\mathscr{O}_X$-module.*

Proof. Nötherian induction on $Y = \overline{\operatorname{Supp} \tilde{M}}$. On $Y = \{\text{point}\}$, clear.

Let $U \subseteq X$ open. Show $\Gamma(X, \tilde{M}) \to \Gamma(U, \tilde{M})$ surjective. WLOG, $Y \cap U \neq \emptyset$. Choose $f \in A$ such that $X_f \subseteq U$ and $X_f \cap Y \neq \emptyset$. Set $Z = V(f) = X \setminus X_f$. Set $I = (f) \subseteq A$.

We know that $\Gamma_I(M)$ is injective, and $\operatorname{Supp}(\Gamma_I(\tilde{M})) \subseteq Y \cap Z$. ($f^n m = 0$ for all $m \in \Gamma_I(M)$, so $f \notin P$ implies $\Gamma_I(M)_P = 0$)

Induction implies that $\widetilde{\Gamma_I(M)}$ is flasque, and $M = \Gamma(X, \tilde{M}) \to \Gamma(U, \tilde{M}) \to \Gamma(X_f, \tilde{M}) = M_f$ gives us $\Gamma(X, \Gamma_I(\tilde{M})) = \Gamma_Z(X, \tilde{M}) \to \Gamma_Z(U, \tilde{M}) = \Gamma(U, \Gamma_I(\tilde{M}))$. Recall that $M \to M_f$ is surjective.

Let $\sigma \in \Gamma(U, \tilde{M})$. Then $\exists \tau' \in \Gamma(X, \tilde{M})$ with the same image in $\Gamma(X_f, \tilde{M})$. Set $\tau = \tau'|_U$. Then $(\sigma - \tau)|_{X_f} = 0$, so $\sigma - \tau \in \Gamma_Z(U, \tilde{M})$.

So $\exists \alpha' \in \Gamma_Z(X, \tilde{M})$ with $\alpha' \mapsto \sigma - \tau$. Therefore, $\alpha' + \tau' \in \Gamma(X, \tilde{M})$ maps to $\sigma \in \Gamma(U, \tilde{M})$. □

Theorem 8.30. *X affine Nötherian scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module, then $H^n(X, \mathscr{F}) = 0$ for all $n > 0$.*

Proof. $X = \operatorname{Spec}(A)$, $\mathscr{F} = \tilde{M}$. A resolution of M by injective A modules gives us a resolution of $\mathscr{F}$ by flasque $\mathscr{O}_X$-modules.

The global section functor gives back the original sequence, so $H^n(X, \tilde{M}) = H^n(I^\cdot)$ which is M if $n = 0$ and 0 otherwise. □

Corollary 8.31. *X Nötherian scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module, then there exists a monomorphism $0 \to \mathscr{F} \to \mathscr{G}$ where $\mathscr{G}$ is a flasque quasicoherent $\mathscr{O}_X$-module.*

Proof. $X = U_1 \cup U_2 \cup \ldots \cup U_n$ where $U_i = \operatorname{Spec} A_i$ where A_i is Nötherian. $\mathscr{F}|_{U_i} = \tilde{M}_i$, M_i an A_i-module. There exists $M_i \subseteq I_i$ for some injective I_i.

$\mathscr{F}_{U_i} \to \tilde{I}_i$ injective, so $\mathscr{F} \to j_*(\tilde{I}_i)$ is injective over U_i. As I_i is injective, $\tilde{I}_i$ is flasque, and $j_*(\tilde{I}_i)$ is flasque.

$0 \to \mathscr{F} \to \oplus_{i=1}^n j_*(\tilde{I}_i)$ is flasque and quasicoherent. □

Exercise: X be a scheme, $f_1, \ldots, f_r \in \Gamma(X, \mathscr{O}_X)$, then if X_{f_i} is affine for all i, and $(f_1, \ldots, f_r) = (1) \subseteq \Gamma(X, \mathscr{O}_X)$, then X is affine.

The ideas is that $A = \Gamma(X, \mathscr{O}_X)$, and we have $\varphi : X \to \operatorname{Spec}(A)$. Then show that $\Gamma(X_{f_i}, \mathscr{O}_X) = A_{f_i}$, and so we get isomorphisms $\varphi : X_{f_i} \to \operatorname{Spec}(A_{f_i})$. So $X = \cup X_{f_i}$ and so φ is a global isomorphism.

Theorem 8.32 (Serre's Criterion)**.** *Let X be a Nötherian scheme. TFAE*

1. *X is affine*
2. *$H^n(X, \mathscr{F}) = 0$ for all quasicoherent $\mathscr{O}_X$-modules $\mathscr{F}$ and $n > 0$.*
3. *$H^1(X, \mathscr{I}) = 0$ for all coherent sheaves of ideals $\mathscr{I} \subseteq \mathscr{O}_X$.*

Proof. 1 implies 2 implies 3 are done already.

Assume 3. Let $P \in X$ be a closed point. Let $P \in U$ be an open affine neighborhood of P. Let $Y = X \setminus U$. We get $0 \to \mathscr{I}_{Y \cup P} \to \mathscr{I}_Y \to k(P) \to 0$, $\Gamma(X, \mathscr{I}_Y) \to \Gamma(X, k(P)) \to H^1(X, \mathscr{I}_{Y \cup P}) = 0$.

$\exists f \in \Gamma(X, \mathscr{I}_Y) \subseteq A = \Gamma(X, \mathscr{O}_X)$ such that $Y \subseteq V(f)$ and $f \mapsto 1 \in k(P)$, so $P \in X_f \subseteq U$, $X_f = U_f$ is affine. Choose $f_1, \ldots, f_r \in A$ such that $X = X_{f_1} \cup \ldots \cup X_{f_r}$ and X_{f_i} is affine. It remains to show that $(f_1, \ldots, f_r) = (1)$.

Claim: $\mathscr{F} \subseteq \mathscr{O}_X^r$ is a coherent subsheaf implies that $H^1(X, \mathscr{F}) = 0$. For $r = 1$ this is three. For $r > 1$, we have $0 \to \mathscr{O}_X^{r-1} \to \mathscr{O}_X^r \to \mathscr{O}_X \to 0$ giving us $0 \to \mathscr{F} \cap \mathscr{O}_X^{r-1} \to \mathscr{F} \to \mathscr{I} \to 0$, and so $H^1(\mathscr{I}) = 0$ and by induction, $H^1(X, \mathscr{F} \cap \mathscr{O}_X^{r-1}) = 0$, so the claim is proved.

Take $\mathscr{O}_X^r \to \mathscr{O}_X \to 0$ by $(g_1, \ldots, g_r) \mapsto \sum g_i f_i$. This map is surjective, and so we take the kernel and call it $\mathscr{F}$, and get a short exact sequence.

$\Gamma(X, \mathscr{O}_X^r) \to \Gamma(X, \mathscr{O}_X) \to H^1(X, \mathscr{F}) = 0$ exact so $(f_1, \ldots, f_r) = (1)$. □

<u>Cech Cohomology</u>

Let X be a topological space and $X = \cup_{i \in I} U_i$. $\mathcal{U} = (U_i)_{i \in I}$, for $i_0, \ldots, i_p \in I$, set $U_{i_0, \ldots, i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$. Let $\mathscr{F}$ be an abelian sheaf on X.

Definition 8.17 (Alternating Function)**.** *A function $\alpha : I^{p+1} \to \{$sections of $\mathscr{F}\}$ is called alternating if $\alpha(i_0, \ldots, i_p) \in \mathscr{F}(U_{i_0, \ldots, i_p})$ and if $\alpha(i_0, \ldots, i_t, i_{t+1}, \ldots, i_p) = -\alpha(i_0, \ldots, i_{t+1}, i_t, \ldots, t_p)$ and if $\alpha(i_1, i_1, i_2, \ldots, i_p) = 0$.*

Definition 8.18 (Cech Complex)**.** *$C^p(\mathcal{U}, \mathscr{F}) = \{$alternating $\alpha : I^{p+1} \to$ sections of $\mathscr{F}\}$ and maps $d : C^p \to C^{p+1}$ by $d\alpha(i_0, \ldots, i_p) = \sum_{k=0}^{p+1} (-1)^k \alpha(i_0, \ldots, \hat{i}_k, \ldots, i_{p+1})|_{U_{i_0, \ldots, i_{p+1}}}$.*

Check: $d^2\alpha = 0$.

Then we have $\check{H}^p(\mathcal{U}, \mathscr{F}) = H^p(C^*(\mathcal{U}, \mathscr{F}))$.

Lemma 8.33. $\check{H}^0(\mathcal{U},\mathscr{F}) = \Gamma(X,\mathscr{F})$.

Proof. $C^0(\mathcal{U},\mathscr{F}) = \prod_{i\in I}\mathscr{F}(U_i)$. If $\alpha \in C^0(\mathcal{U},\mathscr{F})$, then $d\alpha(i,j) = \alpha(j)|_{U_{i,j}} - \alpha(i)|_{U_{i,j}}$, so $d\alpha = 0$ iff the sections are all compatible which is true iff $\exists\sigma \in \mathscr{F}(X)$ such that $\sigma|_{U_i} = \alpha(i)$.

Therefore, $\check{H}^0(\mathcal{U},\mathscr{F}) = \ker(C^0 \to C^1) = \mathscr{F}(X)$. □

Example: $X = \mathbb{P}^1 = \operatorname{Proj} k[x,y]$. $\mathscr{F} = \Omega_{\mathbb{P}^1} = \Omega$. $\mathcal{U} = \{U,V\}$ with $U = D_+(x) = \operatorname{Spec} k[t]$, $t = y/x$ and $V = D_+(y) = k[t^{-1}]$. $U\cap V = \operatorname{Spec} k[t,t^{-1}]$. Then we denote $C^p = C^p(\mathcal{U},\Omega)$.

$C^0 = \Gamma(U,\Omega)\oplus\Gamma(V,\Omega) = k[t]dt \oplus k[t^{-1}]d(t^{-1})$, $C^1 = \Gamma(U\cap V,\Omega) = k[t,t^{-1}]dt$. $C^2 = 0$.

$d : C^0 \to C^1$ by $dt \mapsto -dt$ and $d(t^{-1}) \mapsto -t^{-2}dt$. $f(t)dt \oplus g(t^{-1})d(t^{-1}) \in \ker d$ iff $-f(t) - t^{-2}g(t^{-1}) = 0$ iff $f = g = 0$, so $\check{H}^0(\mathcal{U},\Omega) = \Omega(X) = 0$.

$\operatorname{Im}(d) = \{(-f(t) - t^{-2}g(t^{-1}))dt\} \subseteq k[t,t^{-1}]dt$. This is $\{\sum a_i t^i | a_{-1} = 0\}$, so $\check{H}^1(\mathcal{U},\Omega) \simeq k$ generated by $t^{-1}dt$.

Let $X = \cup_{i\in I}U_i$, and $\mathcal{U} = (U_i)_{i\in I}$. Then we have defined $C^p(\mathcal{U},\mathscr{F})$.

Definition 8.19. *Define an abelian sheaf* $\mathscr{C}^p(\mathcal{U},\mathscr{F}) = \mathscr{C}^p$ *given by* $\Gamma(V,\mathscr{C}^p(\mathcal{U},\mathscr{F})) = C^p(\mathcal{U}, j_*(\mathscr{F}|_V))$ *where* j *is the inclusion* $V \to X$.

$0 \to \mathscr{F} \xrightarrow{\epsilon} \mathscr{C}^0 \xrightarrow{d} \mathscr{C}^1 \to \dots$. $\sigma \in \mathscr{F}(V)$, then $\epsilon\sigma \in C^0(\mathcal{U}, j_*\mathscr{F}|_V)$ and $\epsilon\sigma(i) = \sigma|_{U_i\cap V}$.

Lemma 8.34. *This is an exact sequence of sheaves.*

Proof. Exact at $p = 0$. $0 \to \Gamma(X, j_*(\mathscr{F}|_V)) \to C^0 \to C^1 \to \dots$. Let $p \geq 1$. $x \in X$, choose $k \in I$ with $x \in U_k$. Let $\alpha_x \in \mathscr{C}^p_x$. Then $\alpha \in \mathscr{C}^p(V)$ for $x \in V$. WLOG, $V \subseteq U_k$.

Define $h\alpha \in C^{p-1}$ by $h\alpha(i_0,\dots,i_{p-1}) = \alpha(k,i_0,\dots,i_{p-1}) \in \mathscr{F}(U_{k,i_0,\dots,i_{p-1}} \cap V) = \mathscr{F}(U_{i_0,\dots,i_p}\cap V)$.

Now define $h : \mathscr{C}^p_x \to \mathscr{C}^{p-1}_x$ by $\alpha_x \mapsto (h\alpha)_x$. Check that $(dh + hd)\alpha = \alpha$, so $dh + hd = id - 0$ so id ~ 0 and so $H^p(\mathscr{C}^*_x) = 0$, and so $\mathscr{C}^*$ is exact. □

Proposition 8.35. *If* $\mathscr{F}$ *is flasque, then* $\check{H}^p(\mathcal{U},\mathscr{F}) = 0$ *for* $p > 0$.

Proof. Let $\mathscr{C}^p = \mathscr{C}^p(\mathcal{U},\mathscr{F})$. This is flasque for all p. So $0 \to \mathscr{F} \to \mathscr{C}^0 \to \mathscr{C}^1 \to \dots$ is an exact sequence of flasque sheaves. Taking the global section functo we get C^p, but it is still exact because it is a flasque sequence. So $\check{H}^p(\mathcal{U},\mathscr{F}) = H^p(C^*) = 0$. □

Note: $\mathscr{F}$ is an abelian sheaf, $0 \to \mathscr{F} \to \mathscr{I}^0 \to \dots$ an inejctive resolution, then we get a map $\check{H}^p(\mathcal{U},\mathscr{F}) \to H^p(X,\mathscr{F})$.

Theorem 8.36. *If* X *is Nötherian separated scheme, and* $\mathcal{U} = (U_i)_{i\in I}$ *is an open affine covering, and* $\mathscr{F}$ *is quasi-coherent* $\mathscr{O}_X$*-module, then* $\check{H}^p(\mathcal{U},\mathscr{F}) = H^p(X,\mathscr{F})$.

Proof. $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0$ with $\mathscr{G}$ flasque and quasicoherent, then $\mathscr{R}$ is quasicoherent. If X is seprated, then $U_{i_0,\dots,i_p}$ is affine, so we get $0 \to \mathscr{F}(U_{i_0,\dots,i_p}) \to \mathscr{G}(U_{i_0,\dots,i_p}) \to \mathscr{R}(U_{i_0,\dots,i_p}) \to 0$.

So we get the long exact sequence $0 \to \check{H}^0(\mathscr{F}) \to \check{H}^0(\mathscr{G}) \to \check{H}^0(\mathscr{R}) \to \check{H}^1(\mathscr{F}) \to 0$ and an exact sequence $H^0(\mathscr{F}) \to H^0(\mathscr{G}) \to H^0(\mathscr{R}) \to H^1(\mathscr{F}) \to 0$, and so taking the vertical maps, for $p = 1$, this proves it.

We can proceed by induction obtaining $0 \to \check{H}^p(\mathscr{R}) \to \check{H}^{p+1}(\mathscr{F}) \to 0$ and $0 \to H^p(\mathscr{R}) \to H^p(\mathscr{F}) \to 0$ and looking at the vertical maps, whcih must be isomorphisms. □

Let A be a Nötherian ring, $S = A[x_0, \dots, x_r]$, and $X = \operatorname{Proj} S = \mathbb{P}^r_A$.

Then let $\mathscr{F}$ be any $\mathscr{O}_X$ module. Recall that $\Gamma_*(\mathscr{F}) = \oplus_{n\in\mathbb{Z}}\Gamma(X, \mathscr{F}(n))$.

Note: $f \in S_m = \Gamma(X, \mathscr{O}(m))$ gives $f : \mathscr{F}(n) \to \mathscr{F}(n+m)$ by $\sigma \mapsto \sigma \otimes f$.

Thus, $f : H^p(X, \mathscr{F}(n)) \to H^p(X, \mathscr{F}(n+m))$, so $\Gamma_*(\mathscr{F})$ is a graded S-module.

Theorem 8.37. *1. $S \to \Gamma_*(\mathscr{O}_X)$ is an isomorphism of graded S-modules.*

2. *$H^p(X, \mathscr{O}_X(n)) = 0$ for $0 < p < r$ any n.*
3. *$H^r(X, \mathscr{O}_X(-r-1)) \simeq A$*
4. *Perfect Pairing: $H^0(X, \mathscr{O}_X(n)) \times H^r(X, \mathscr{O}_X(-n-r-1)) \to H^r(X, \mathscr{O}_X(-r-1)) = A$.*

Proof. $\mathscr{F} = \oplus_{n\in\mathbb{Z}}\mathscr{O}_X(n)$ is quasicoherent. Then $H^p(X, \mathscr{F}) = \oplus_{n\in\mathbb{Z}}H^p(X, \mathscr{O}_X(n))$. Let $\mathcal{U} = \{U_0, \dots, U_r\}$ with $U_i = D_+(x_i) \subseteq X$. Then $U_{i_0,\dots,i_p} = D_+(x_{i_0}\dots x_{i_p})$, so we look at $\Gamma(U_{i_0,\dots,i_p}, \mathscr{O}_X(n)) = S(n)_{(x_{i_0}\dots x_{i_p})}$, and so $\Gamma(U_{i_0,\dots,i_p}, \mathscr{F}) = S_{x_{i_0}\dots x_{i_p}}$.

$C^*(\mathcal{U}, \mathscr{F})$: $\oplus_{i_0} S_{x_{i_0}} \to \oplus_{i_0<i_1} S_{x_{i_0}x_{i_1}} \to \dots \to \oplus_{k=0}^r S_{x_0\dots\hat{x}_k\dots x_r} \to S_{x_0\dots x_r}$

1. ker(first map) $= \cap S_{x_i} = S$.
2. $C^p(\mathcal{U}, \mathscr{F})_{x_r} = \prod_{i_0<\dots<i_p} S_{x_{i_0}\dots x_{i_p}x_{i_r}} = \prod_{i_0<\dots<i_p} \oplus_{n\in\mathbb{Z}}\Gamma(U_{i_0,\dots,i_p}\cap U_r, \mathscr{O}_X(n))$.

   This equals $\prod \Gamma(U_{i_0,\dots,i_p} \cap U_r, \mathscr{F}) = C^p(\mathcal{U}', \mathscr{F}|_{U_r})$ where $\mathcal{U}' = \{U_i \cap U_r\}$ is an affine open covering of U_r. Thus, $\check{H}^p(\mathcal{U}, \mathscr{F})_{x_r} = \check{H}^p(\mathcal{U}', \mathscr{F}|_{U_r}) = H^p(U_r, \mathscr{F}|_{U_r}) = 0$. So every element of $H^p(X, \mathscr{F})$ is killed by x_r^N. It is enough to show that $H^p(X, \mathscr{F}) \xrightarrow{x_r} H^p(X, \mathscr{F})$ is an isomorphism. WLOG, $r \geq 2$.

   $H = V(x_r) \subseteq X$, $H \simeq \mathbb{P}^{r-1}_A$. We have $0 \to \mathscr{O}_X(-1) \to \mathscr{O}_X \to \mathscr{O}_H \to 0$. As $\mathscr{F}$ is locally free, tensoring with it is flat, so we get $0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{F}_H \to 0$, and $\mathscr{F}_H = \oplus_{n\in\mathbb{Z}}\mathscr{O}_H(n)$, with the first map being multiplication by x_r. By induction on r, we get that $H^p(X, \mathscr{F}_H) = H^p(H, \mathscr{F}_H) = 0$ for $0 < p < r-1$.

   So we get a long exact sequence $H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}_H) \to H^1(X, \mathscr{F}(-1)) \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{F}_H)$, so we get that $H^1(X, \mathscr{F}) \simeq H^1(X, \mathscr{F}(-1))$.

For $1 < p < r-1$, we have $H^{p-1}(X, \mathscr{F}_H) = 0 \to H^p(X, \mathscr{F}(-1)) \simeq H^p(X, \mathscr{F}) \to H^p(X, \mathscr{F}_H) = 0$.

So we now have $H^{r-2}(X, \mathscr{F}_H) \xrightarrow{0} H^{r-1}(X, \mathscr{F}(-1)) \xrightarrow{x_r} H^{r-1}(X, \mathscr{F}) \to H^{r-1}(X, \mathscr{F}_H) \xrightarrow{\delta} H^r(X, \mathscr{F}(-1)) \xrightarrow{x_r} H^r(X, \mathscr{F}) \to 0$. Note that if $r = 2$, we don't get $H^{r-2}(X, \mathscr{F}_H) = 0$, but the first map is zero.

$\mathrm{Ann}(x_r) \subseteq H^r(X, \mathscr{F}(-1))$ has basis $\{x_0^{\ell_0} \dots x_{r-1}^{\ell_{r-1}} x_r^{-1} | \ell_i < 0\}$ and $H^{r-1}(X, \mathscr{F}_H)$ has basis $\{x_0^{\ell_0} \dots x_{r-1}^{\ell_{r-1}} | \ell_i < 0\}$. T his tells us that δ is injective, and so the map before it is zero and we have an isomorphism $H^{r-1}(X, \mathscr{F}(-1)) = H^{r-1}(X, \mathscr{F})$.

3. Image(last map)= $\mathrm{span}_A\{x_0^{\ell_0} \dots x_r^{\ell_r} | \ell_i \in \mathbb{Z}$ with some $\ell_i \geq 0\}$, so we get $H^r(X, \mathscr{F}) = \mathrm{span}_A\{x_0^{\ell_0} \dots x_r^{\ell_r} | \ell_i < 0 \forall i\}$. And so $H^r(X, \mathscr{O}_X(-r-1)) = A(x_0 x_1 \dots x_r)^{-1} \simeq A$.

4. $H^r(X, \mathscr{O}_X(-n-r-1)) = \mathrm{span}_A\{x_0^{\ell_0} \dots x_r^{\ell_r} | \sum \ell_i = -n-r-1, \ell_i < 0\}$. So look at $H^0(X, \mathscr{O}_X(n)) \times H^r(X, \mathscr{O}_X(-n-r-1)) \to H^r(X, \mathscr{O}_X(-r-1))$. map $(x_0^{m_0} \dots x_r^{m_r}) \times (x_0^{\ell_0} \dots x_r^{\ell_r}) \mapsto x_0^{\ell_0+m_0} \dots x_r^{\ell_r+m_r}$.

   The right hand side $\neq 0 \in H^r(X, \mathscr{O}_X(-r-1))$ iff $m_i + \ell_i = -1$ for all i, so $\ell_i = -m_i - 1$ for all i.

□

Theorem 8.38. *X a projective scheme over* $\mathrm{Spec}\, A$*, A Nötherian and $\mathscr{O}_X(1)$ is a very ample invertible sheaf relative to* $\mathrm{Spec}(A)$*. If $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, then*

1. *$H^p(X, \mathscr{F})$ is a finitely generated A-module for all $p \geq 0$.*
2. *$H^p(X, \mathscr{F}(n)) = 0$ for all $p > 0$ for $n >> 0$.*

Proof. Let $f : X \to \mathbb{P}^r$, $\mathscr{O}_X(1) = f^*\mathscr{O}_{\mathbb{P}^r}(1)$. $f_*\mathscr{F}$ is coherent on $\mathbb{P}^r$. $f_*(\mathscr{F}(n)) = f_*(f^*\mathscr{O}_{\mathbb{P}^r}(n) \otimes \mathscr{F}) = \mathscr{O}_{\mathbb{P}^r}(n) \otimes f_*\mathscr{F} = (f_*\mathscr{F})(n)$.

$H^p(X, \mathscr{F}(n)) = H^p(\mathbb{P}^r, f_*\mathscr{F}(n)) = H^p(\mathbb{P}^r, (f_*\mathscr{F})(n))$.

WLOG, $X = \mathbb{P}^r$. Note: Theorem is true for $\mathscr{F} = \mathscr{O}_{\mathbb{P}^r}(n)$.

The theorem is true for $p > r$, as all the cohomology vanishes. We proceed by decreasing induction on p. There exists $\mathscr{E} = \oplus \mathscr{O}_{\mathbb{P}^r}(q_i)$ with finitely many terms with $0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0$ exact, so $0 \to \mathscr{R}(n) \to \mathscr{E}(n) \to \mathscr{F}(n) \to 0$ exact.

Thus, $H^p(\mathbb{P}^r, \mathscr{E}(n)) \to H^p(\mathbb{P}^r, \mathscr{F}(n)) \to H^{p+1}(\mathbb{P}^r, \mathscr{R}(n))$. The outer modules are finitely generated by the previous theorem and inductions. As A is Nötherian, the middle module is finitely generated.

If the outer modules are zero, then $H^p(\mathbb{P}^r, \mathscr{F}(n)) = 0$ for all $n >> 0$, $p > 0$. □

Application: if X is a nonsingular curve, defined genus to be $g = \dim_k \Gamma(X, \Omega_X) < \infty$. With Serre duality (to be shown), we see that $g = \dim_k H^1(X, \mathscr{O}_X)$

Euler Characteristic

Let X be a projective scheme over k. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module and $\dim(X) = r$.

Definition 8.20 (Euler Characteristic). *The Euler characteristic of $\mathscr{F}$ is $\chi(\mathscr{F}) = \sum_{p\geq 0}(-1)^p \dim_k H^p(X, \mathscr{F})$.*

Then $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ gives a long exact sequence $0 \to H^0(X, \mathscr{F}') \to H^0(X, \mathscr{F}) \to \ldots \to H^r(X, \mathscr{F}) \to H^r(X, \mathscr{F}'') \to 0$ an exact sequence of k-vector spaces.

So $\chi(\mathscr{F}) = \chi(\mathscr{F}')+\chi(\mathscr{F}'')$. In general, if we have $0 \to \mathscr{F}^0 \to \ldots \to \mathscr{F}^\ell = 0$, then $\sum(-1)^i\chi(\mathscr{F}^i) = 0$.

Proposition 8.39. *X a proper scheme over Nötherian* Spec(A). *$\mathscr{L}$ an invertible $\mathscr{O}_X$-module. Then $\mathscr{L}$ is ample iff $\forall$ coherent $\mathscr{O}_X$-modules $\mathscr{F}$, we have $H^p(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $p > 0$, $n >> 0$.*

Proof. $\Rightarrow$: $\mathscr{L}$ ample, then $\mathscr{L}^{\otimes m}$ is very ample relative to $\operatorname{Spec} A$ for some $m > 0$. Thus, there exists an immersion $X \subseteq X' \subseteq \mathbb{P}^r_A$ with $\mathscr{L}^{\otimes m} = \mathscr{O}_X(1)$. X is proper, so $X = X'$ is closed in $\mathbb{P}^r_A$. Choose $n_0 > 0$ such that $H^p(X, \mathscr{F} \otimes \mathscr{L}^i \otimes \mathscr{O}_X(n)) = 0$ for all $p > 0$ and for $0 \leq i < m$ and $n \geq n_0$. Thus, $H^p(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for $n \geq m + mn_0$.

$\Leftarrow$: $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Show: $\mathscr{F} \otimes \mathscr{L}^n$ generated by global sections for all $n >> 0$. Let $p \in X$ be a closed point. Then we get $0 \to \mathscr{I}_p \to \mathscr{O}_X \to k(P) \to 0$ ses, so we have $\mathscr{I}_p \otimes \mathscr{F} \to \mathscr{F} \to \mathscr{F} \otimes k(p) \to 0$, and to get a short exact sequence, we take $0 \to \mathscr{I}_p\mathscr{F} \to \mathscr{F} \to \mathscr{F} \otimes k(P) \to 0$.

So then we have $0 \to \mathscr{I}_p\mathscr{F} \otimes \mathscr{L}^n \to \mathscr{F} \otimes \mathscr{L}^n \to \mathscr{F} \otimes \mathscr{L}^n \otimes k(p) \to 0$. For $n \geq n_0 = n_0(p)$, we get $\Gamma(X, \mathscr{F}\otimes\mathscr{L}^n) \to \Gamma(X, \mathscr{F}\otimes\mathscr{L}^n\otimes k(P)) \to H^1(X, \mathscr{I}_p\mathscr{F}\otimes \mathscr{L}^n) = 0$, so the second term is $(\mathscr{F} \otimes \mathscr{L}^n)_p \otimes k(p)$. By Nakayama, $(\mathscr{F} \otimes \mathscr{L}^n)_p$ is generated by global sections from $\Gamma(X, \mathscr{F} \otimes \mathscr{L}^n)$. $p \in U = \operatorname{Spec}(B) \subseteq X$, B Nötherian.

So $M = \Gamma(U, \mathscr{F} \otimes \mathscr{L}^n)$ is a finitely generated B-module. $M' \subseteq M$ is the submodule generated by the imeage of $\Gamma(X, \mathscr{F} \otimes \mathscr{L}^n)$. $M_p = \mathscr{F}_p = M'_p$. Then there exists $f \in B \setminus p$ suc that $M'_f = M_f$.

Set $U_{p,n} = D(f) \subseteq U$, then $p \in U_{p,n}$ and $(\mathscr{F} \otimes \mathscr{L}^n)|_{U_{p,n}}$ is generated by global sections.

Special case: may choose n_1 and open $V_p \ni p$ such that $\mathscr{L}^{n_1}|_{V_p}$ is generated by global sections. Set $U_p = V_p \cap U_{p,n_0} \cap \ldots \cap U_{p,n_0+n_1-1}$. Then if $n \geq n_0(p)$, we have $\mathscr{F}\otimes\mathscr{L}^n = (\mathscr{F}\otimes\mathscr{L}^{n'})\otimes(\mathscr{L}^{n_1})^m$ where $n_0 \leq n' < n_0+n_1$ with $m \geq 0$. Then $(\mathscr{F} \otimes \mathscr{L}^n)|_{U_p}$ is generated by global sections from $\Gamma(X, \mathscr{F} \otimes \mathscr{L}^n)$.

So $X = U_{p_1} \cup U_{p_2} \cup \ldots \cup U_{p_\ell}$ with $n \geq \max(n_0(p_1), \ldots, n_0(p_\ell))$, and we get $\mathscr{F} \otimes \mathscr{L}^n$ generated by global sections. □

Applications: A Nötherian, $X = \mathbb{P}^r_A = \operatorname{Proj}(S)$ for $S = A[x_0, \ldots, x_r]$. Let M be a finitely generated graded S-module. We have maps $M_n \to \Gamma(\mathbb{P}^r, \tilde{M}(n))$, a homomorphism of graded S-modules $\varphi : M \to \Gamma_*(\tilde{M}) = \oplus_{n\in\mathbb{Z}}\Gamma(\tilde{M}(n))$.

Claim: $M_n \simeq \Gamma(\mathbb{P}^r, \tilde{M}(n))$ for all $n >> 0$

Injective: $N = \ker\varphi \subseteq M$. Show that $N_n = 0$ for $n >> 0$. As N is finitely generated, it is enough to show that $\forall m \in N$ and $0 \le i \le r$, there exists $p > 0$ so that $x_i^p m = 0$. This is true because $m = 0$ in $\Gamma(D_+(x_i), \tilde{M}(n)) = M(n)_{(x_i)} \subseteq M_{x_i}$.

Surjective: $0 \to F' \to F \to M \to 0$, $F = \oplus S(q_i)$ finite sum. So then $0 \to \tilde{F}'(n) \to \tilde{F}(n) \to \tilde{M}(n) \to 0$, and so for $n >> 0$, we get $\Gamma(\mathbb{P}^r, \tilde{F}(n)) \to \Gamma(\mathbb{P}^r, \tilde{M}(n)) \to H^1(\mathbb{P}^r, \tilde{F}'(n)) = 0$, so we have a surjection, but this is really an onto map $F_n \to M_n$, and must also give a surejction $M_n \to \Gamma(\mathbb{P}^r, \tilde{M}(n))$.

X is projective over a field, k. $X \subseteq \mathbb{P}^r_k$. Then there is $\mathscr{O}_X(1)$ very ample. $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module.

Definition 8.21 (Hilbert Polynomial)**.** *We define the hilbert polynomial to be* $P_{\mathscr{F}}(n) = \chi(\mathscr{F}(n))$*. In particular,* $P_X(n) = P_{\mathscr{O}_X}(n)$*.*

Claim: $P_{\mathscr{F}}(n) \in \mathbb{Q}[n]$.

Proof. Nötherian induction on $Y = \overline{\operatorname{Supp}\mathscr{F}}$. $Y = \emptyset \Rightarrow \mathscr{F} = 0 \Rightarrow P_{\mathscr{F}}(n) = 0$.

If $Y \neq \emptyset$, take $f \in \Gamma(X, \mathscr{O}_X(1))$ such that $Y \cap X_f \neq 0$. $0 \to \mathscr{R} \to \mathscr{F}(-1) \xrightarrow{f\cdot} \mathscr{F} \to \mathscr{M} \to 0$ is an exact sequence of coherent $\mathscr{O}_X$-modules.

Taking the Euler characteristics at n, we get $P_{\mathscr{F}}(n) - P_{\mathscr{F}}(n-1) = P_{\mathscr{M}}(n) - P_{\mathscr{R}}(n)$. The closures of $\operatorname{Supp}\mathscr{R}$ and $\operatorname{Supp}\mathscr{M} \subseteq Y \cap V(f) \subsetneq Y$. By Nötherian induction, the functions on the right hand side are polynomials in $\mathbb{Q}[n]$, and therefore $P_{\mathscr{F}}(n) \in \mathbb{Q}[n]$. $\square$

Note $\mathbb{P}^r = \operatorname{Proj} S$, $S = k[x_0, \ldots, x_r]$, M is a finitely generated graded S-module. If $n >> 0$, then $M_n = \Gamma(\mathbb{P}^r, \tilde{M}(n)) = \chi(\tilde{M}(n))$.

Note: $P_{\mathscr{F}}(n)$ depends on $\mathscr{O}_X(1)$ on $\mathscr{O}_X(1)$, but $P_{\mathscr{F}}(0) = \chi(\mathscr{F})$. Assume $\dim X = r$. Then the arithmetic genus if $p_a(X) = (-1)^r(\chi(\mathscr{O}_X) - 1) = (-1)^r(P_X(0) - 1)$, so if X is a connected curve, then $p_a(X) = 1 - \chi(\mathscr{O}_X) = 1 - \dim_k H^0(X, \mathscr{O}_X) + \dim_k(H^1(X, \mathscr{O}_X))$. So then, as X is projective, this is $\dim_k H^1(X, \mathscr{O}_X)$.

Ext Groups and Sheaves

$(X, \mathscr{O}_X)$ a ringed space, $\mathscr{F}, \mathscr{G}$ are $\mathscr{O}_X$-modules. Remember that $\hom(\mathscr{F}, -)$ is a functor from $Mod(X) \to Ab$ is left exact.

Definition 8.22 (Ext)**.** $\operatorname{Ext}^p(\mathscr{F}, -) = R^p \hom(\mathscr{F}, -)$ *from* $Mod(X) \to Ab$*.*
$\mathscr{E}xt^p(\mathscr{F}, -) = R^p \mathscr{H}om(\mathscr{F}, -)$.

Proposition 8.40. *1.* $\mathscr{E}xt^0(\mathscr{O}_X, \mathscr{G}) = \mathscr{G}$*,* $\mathscr{E}xt^p(\mathscr{O}_X, \mathscr{G}) = 0$ *for* $p > 0$*.*

2. $\operatorname{Ext}^p(\mathscr{O}_X, \mathscr{G}) = H^p(X, \mathscr{G})$

Lemma 8.41. $\mathscr{I} \in Mod(X)$ *injective,* $U \subseteq X$ *open, then* $\mathscr{I}|_U \in Mod(U)$ *injective.*

Proof. $j : U \to X$ inclusion.

$$\begin{array}{ccc} j_*\mathscr{F} & \longrightarrow & j_!\mathscr{G} \\ \big\downarrow & & \big\downarrow \\ j_*(\mathscr{I}|_U) & \xrightarrow{\subset} & \mathscr{I} \end{array}$$

□

Proposition 8.42. *$U \subseteq X$ open. Then $\mathscr{E}xt^p_X(\mathscr{F},\mathscr{G})|_U = \mathscr{E}xt^p_U(\mathscr{F}|_U,\mathscr{G}|_U)$.*

Note: $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ gives a long exact sequence on $\mathrm{Ext}(\mathscr{F},-)$

Proposition 8.43. *$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ gives a long exact sequence $0 \to \hom(\mathscr{F}'',\mathscr{G}) \to \hom(\mathscr{F},\mathscr{G}) \to \hom(\mathscr{F}'',\mathscr{G}) \to \mathrm{Ext}^1(\mathscr{F}'',\mathscr{G}) \to \dots$.*

Proof. $0 \to \mathscr{G} \to \mathscr{I}^0 \to \dots$. Then as $\hom(-,\mathscr{I}^P)$ is exact and contravariant, we get $0 \to \hom(\mathscr{F}'',\mathscr{I}^*) \to \hom(\mathscr{F},\mathscr{I}^*) \to \hom(\mathscr{F}',\mathscr{I}^*) \to 0$, and so we get a long exact sequence in the first variable. □

Definition 8.23 (Locally Free Resolution)**.** *A locally free resolution of $\mathscr{F}$ is a resolution $\dots \to \mathscr{E}_2 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$ exact with $\mathscr{E}_p$ locally free of finite rank.*

Proposition 8.44. *If $\mathscr{E}_* \to \mathscr{F} \to 0$ is locally free resolution, then $\mathscr{E}xt^p(\mathscr{F},\mathscr{G}) = H^p(\mathscr{H}om(\mathscr{E}_*,\mathscr{G}))$.*

Proof. Both sides are δ-functors. RHS: $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$, $\mathscr{E}_p$ is locally free implies that $\mathscr{H}om(\mathscr{E}_p,-)$ is exact, and so we get $0 \to \mathscr{H}om(\mathscr{E}_*,\mathscr{G}'') \to \mathscr{H}om(\mathscr{E}_*,\mathscr{G}) \to \mathscr{H}om(\mathscr{E}_*,\mathscr{G}') \to 0$, so get long exact sequence.

Agree for $p = 0$: $\mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$, so $0 \to \mathscr{H}om(\mathscr{F},\mathscr{G}) \to \mathscr{H}om(\mathscr{E}_0,\mathscr{G}) \to \mathscr{H}om(\mathscr{E}_1,\mathscr{G}) \to \dots$ is exact, and so $H^0(\mathscr{H}om(\mathscr{E}_*,\mathscr{G})) = \mathscr{H}om(\mathscr{F},\mathscr{H}) = \mathscr{E}xt^0(\mathscr{F},\mathscr{G})$.

Universality: Both sides vanish when $\mathscr{G}$ is injective. □

Note: $\mathscr{E}$ is locally free $\mathscr{O}_X$-module of finite rank, then $\mathscr{E}^\vee = \mathscr{H}om(\mathscr{E},\mathscr{O}_X)$is locally free.

Exercise: $\mathscr{H}om(\mathscr{F},\mathscr{G}) \otimes \mathscr{E} \simeq \mathscr{H}om(\mathscr{F},\mathscr{E} \otimes \mathscr{G}) \simeq \mathscr{H}om(\mathscr{F} \otimes \mathscr{E}^\vee,\mathscr{G})$ by $\varphi \otimes e \mapsto [f \mapsto e \otimes \varphi(f)]$ and $\psi \mapsto [f \otimes e^\vee \mapsto (e^\vee \otimes 1)(\psi(f))]$

Special Case: $\mathscr{E} = \mathscr{H}om(\mathscr{O},\mathscr{E}) = \mathscr{H}om(\mathscr{E}^\vee,\mathscr{O}_X) = \mathscr{E}^{\vee\vee}$.

Lemma 8.45. *$\mathscr{I}$ is an injective $\mathscr{O}_X$-module, $\mathscr{E}$ is a locally free $\mathscr{O}_X$-module, then $\mathscr{I} \otimes \mathscr{E}$ is injective.*

Proof. $\hom(-,\mathscr{I} \otimes \mathscr{E}) = \hom(- \otimes \mathscr{E}^\vee,\mathscr{I})$ is exact. □

Proposition 8.46. *$\mathscr{E}$ locally free of finite rank, then $\mathrm{Ext}^p(\mathscr{F},\mathscr{E}\otimes\mathscr{G}) = \mathrm{Ext}^p(\mathscr{F}\otimes\mathscr{E}^\vee,\mathscr{G})$ and $\mathscr{E}xt^p(\mathscr{F},\mathscr{G}) \otimes \mathscr{E} = \mathscr{E}xt^p(\mathscr{F},\mathscr{E} \otimes \mathscr{G}) = \mathscr{E}xt^p(\mathscr{F} \otimes \mathscr{E}^\vee,\mathscr{G})$.*

Proof. Everything is a δ-functor. They are the same for $p = 0$.

They vanish for $\mathscr{G}$ injective. □

Note: If A is a ring, M an A-module, then $\operatorname{Ext}^p_A(M,-) = R^p \hom_A(M,-)$. $(\{pt\}, A)$ is a ringed space, and modules on it and A-mod are the same thing. If $\mathscr{E}$ is locally free of finite rank, then $\mathscr{H}om(\mathscr{E},\mathscr{F})_x = \hom_{\mathscr{O}_{X,x}}(\mathscr{E}_x,\mathscr{F}_x)$.

Proposition 8.47. *X a Nötherian Scheme, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module and $\mathscr{G}$ any $\mathscr{O}_X$-module. Then $\mathscr{E}xt^p(\mathscr{F},\mathscr{G})_x = \operatorname{Ext}^p_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$ for all $x \in X, p \geq 0$.*

Proof. WLOG, X is affine. There exists a locally free resolution $\mathscr{E}_* \to \mathscr{F} \to 0$, so there exists a free resolution of $\mathscr{O}_{X,x}$-modules $\mathscr{E}_{*,x} \to \mathscr{F}_x \to 0$.

$\mathscr{E}xt^p(\mathscr{F},\mathscr{G})_x = H^p(\mathscr{H}om(\mathscr{E}_*,\mathscr{G}))_x = H^p(\mathscr{H}om(\mathscr{E}_*,\mathscr{G})_x) = H^p(\hom(\mathscr{E}_{*,x},\mathscr{G}_x)) = \operatorname{Ext}^p_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$. □

Proposition 8.48. *X a projective scheme over Nötherian $\operatorname{Spec}(A)$ with $\mathscr{O}_X(1)$ is very ample relative to A. $\mathscr{F},\mathscr{G}$ coherent $\mathscr{O}_X$-modules, $p \geq 0$. Then $\operatorname{Ext}^p(\mathscr{F},\mathscr{G}(n)) = \Gamma(X, \mathscr{E}xt^p(\mathscr{F},\mathscr{G}(n)))$ for all $n >> 0$.*

Proof. $p=0$: $\hom(\mathscr{F},\mathscr{G}(n)) = \Gamma(X,\mathscr{H}om(\mathscr{F},\mathscr{G}(n)))$ is true for all n.

$\mathscr{F}$ is lcoally free of finite rank. $\operatorname{Ext}^p(\mathscr{F},\mathscr{G}(n)) = \operatorname{Ext}^p(\mathscr{O}_X,\mathscr{F}^\vee \otimes \mathscr{G}(n)) = H^p(X,\mathscr{F}^\vee \otimes \mathscr{G}(n)) = 0$. $\mathscr{E}xt^p(\mathscr{F},\mathscr{G}(n)) = \mathscr{E}xt^p(\mathscr{O}_X,\mathscr{F}^\vee \otimes \mathscr{G}(n)) = 0$ for all $p > 0$ for all n, so it is ture for all $n >> 0$.

$\mathscr{F}$ coherent: $0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0$ where $\mathscr{E}$ is locally free. $0 \to \mathscr{H}om(\mathscr{F},\mathscr{G}(n)) \to \mathscr{H}om(\mathscr{E},\mathscr{G}(n)) \to \mathscr{H}om(\mathscr{R},\mathscr{G}(n)) \to \mathscr{E}xt^1(\mathscr{F},\mathscr{G}(n)) \to \mathscr{E}xt^1(\mathscr{E},\mathscr{G}(n)) = 0$ and we can take the twisting outside.

So for $n >> 0$, we have $0 \to \hom(\mathscr{F},\mathscr{G}(n)) \to \hom(\mathscr{E},\mathscr{G}(n)) \to \hom(\mathscr{R},\mathscr{G}(n)) \to \Gamma(X,\mathscr{E}xt^1(\mathscr{F},\mathscr{G}(n))) \to 0$.

We also get $0 \to \hom(\mathscr{F},\mathscr{G}(n)) \to \hom(\mathscr{E},\mathscr{G}(n)) \to \hom(\mathscr{R},\mathscr{G}(n)) \to \operatorname{Ext}^1(\mathscr{F},\mathscr{G}(n)) \to \operatorname{Ext}^1(\mathscr{E},\mathscr{G}(n)) = 0$ for $n >> 0$. As the first three are the same, the last one must be.

For $p \geq 2$, we have long exact $0 \to \operatorname{Ext}^{p-1}(\mathscr{R},\mathscr{G}(n)) \to \operatorname{Ext}^p(\mathscr{F},\mathscr{G}(n)) \to 0$ for $n >> 0$. Also, for all n, $0 \to \mathscr{E}xt^{p-1}(\mathscr{R},\mathscr{G}(n)) \to \mathscr{E}xt^p(\mathscr{F},\mathscr{G}(n)) \to 0$.

By induction, we get vertical maps, which prove the theorem. □

Let A be a ring, M an A-module, $\bigwedge^p M = M \otimes \ldots \otimes M / < \ldots x \otimes x \ldots >$.

$0 \to M \to N \to R \to 0$ a ses of free modules of ranks m,n,r, then $\bigwedge^n N \simeq \bigwedge^m M \otimes_A \bigwedge^r R$ is a natural isomorphism. This is important for sheafification.

Let $\mathscr{F}$ be an $\mathscr{O}_X$-module, $\bigwedge^p \mathscr{F} = (U \mapsto \bigwedge^p \mathscr{F}(U))^+$.

Then if $0 \to \mathscr{M} \to \mathscr{N} \to \mathscr{R} \to 0$ a short exact sequence of locally free $\mathscr{O}_X$-modules of ranks m,n,r, then $\bigwedge^n \mathscr{N} \simeq \bigwedge^m \mathscr{M} \otimes \bigwedge^r \mathscr{R}$.

Let X be a nonsingular variety over a field k. Then Ω_X is locally free of rank n.

Definition 8.24 (Canonical Sheaf of X)**.** *The canonical sheaf on X is $\omega_X = \bigwedge^n \Omega_X$.*

On $\mathbb{P}^n_k$, we have $0 \to \Omega_{\mathbb{P}^n} \to \mathscr{O}(-1)^{\oplus n+1} \to \mathscr{O}_{\mathbb{P}^n} \to 0$. So $\omega_{\mathbb{P}^n} = \bigwedge \Omega_{\mathbb{P}^n} = \bigwedge^n \Omega_{\mathbb{P}^n} \otimes \bigwedge^1 \mathscr{O}_{\mathbb{P}^n} = \bigwedge^{n+1}(\mathscr{O}(-1)^{\oplus n+1}) = \mathscr{O}(-1)^{\otimes n+1} = \mathscr{O}(-n-1)$.

Theorem 8.49 (Serre Duality, Version I)**.** *$X = \mathbb{P}^n_k$, k a field.*

1. $H^n(X, \omega_X) \simeq k$.

2. $\mathscr{F}$ *coherent* $\mathscr{O}_X$*-module implies that* $\hom(\mathscr{F}, \omega_X) \times H^n(X, \mathscr{F}) \to H^n(X, \omega_X) = k$ *is a perfect pairing.*

3. $\operatorname{Ext}^p(\mathscr{F}, \omega_X) \simeq H^{n-p}(X, \mathscr{F})^*$*, a natural isomorphism of functors in* $\mathscr{F}$ *from Coherent Sheaves on* X *to* k*-modules.*

   (If V *is a* k*-vector space,* $V^* = \hom_k(V, H^n(X, \omega_X))$*)*

Proof. 1. $H^n(X, \omega_X) = H^n(X, \mathscr{O}(-n-1)) \simeq k$.

2. Let $\mathscr{F} = \mathscr{O}(q)$. Then $\hom(\mathscr{F}, \omega_X) \times H^n(X, \mathscr{F}) = \Gamma(X, \mathscr{O}(-n-1-q)) \times H^n(X, \mathscr{O}(q)) \to H^n(X, \mathscr{O}(-n-1)) = k$ is a perfect pairing. So now consider the case where $\mathscr{F}$ is coherent. Then $\exists \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$ exact, $\mathscr{E}_i = \oplus \mathscr{O}(q_{ij})$ a finite sum. Note that $H^n(X, -)$ is right exact (end of the long exact sequence). So we get $H^n(X, \mathscr{E}_1) \to H^n(X, \mathscr{E}_0) \to H^n(X, \mathscr{F}) \to 0$, which gives us $0 \to H^n(X, \mathscr{F})^* \to H^n(X, \mathscr{E}_0)^* \to H^n(X, \mathscr{E}_1)^*$. We also have $0 \to \hom(\mathscr{F}, \omega_X) \to \hom(\mathscr{E}_0, \omega_X) \to \hom(\mathscr{E}_1, \omega_X)$, with the last two on each being equal. Thus, $\hom(\mathscr{F}, \omega_X) = H^n(X, \mathscr{F})^*$.

3. Both sides are contravariant δ-functors from $Coh(X) \to Mod(k)$, ie δ-functors $Coh(X)^{op} \to Mod(k)$. They agree for $p = 0$. It remains to show that they are universal: we will show that both sides are erasable. That is, for all coherent $\mathscr{F}$, there exists an epi $u : \mathscr{E} \to \mathscr{F}$ such that both functors vanish on u when $p > 0$.

   So there is an epi $\mathscr{E} \to \mathscr{F}$ when $\mathscr{E}$ is a finite $\oplus \mathscr{O}(q_i)$. WLOG, $q_i < 0$ as $\mathscr{O}(q-1)^{\oplus n+1} \to \mathscr{O}(q)$. So $\operatorname{Ext}^p(\mathscr{E}, \omega_X) = \oplus \operatorname{Ext}^p(\mathscr{O}(q_i), \mathscr{O}(-n-1)) = H^p(X, \mathscr{O}_X(-n-1-q_i)) = 0$ for $p > 0$. Also, $H^{n-p}(X, \mathscr{E}) = \oplus H^{n-p}(X, \mathscr{O}(q_i)) = 0$ for $p > 0$.

□

Definition 8.25 (Dualizing Sheaf)**.** *Let* X *be a proper scheme of dimension* n *over a field* k*. A dualizing sheaf for* X *is a coherent* $\mathscr{O}_X$*-module* ω_X° *together with a* k*-linear trace map* $t : H^n(X, \omega_X^\circ) \to k$ *such that forall coherent* $\mathscr{F}$ *we get perfect pairings* $\hom(\mathscr{F}, \omega_X^\circ) \times H^n(X, \mathscr{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{t} k$.

Proposition 8.50. *Let* (ω, t) *and* (ω', t') *be dualizing sheaves for* X*. Then there exists a unique isomorphism* $\varphi : \omega \to \omega'$ *such that* $t = t' \circ \varphi_* : H^n(X, \omega) \to H^n(X, \omega') \to k$*.*

Proof. The perfect pairing requirement gives $\hom(\omega, \omega') \times H^n(X, \omega) \to H^n(X, \omega') \xrightarrow{t'} k$, and so $t : H^n(X, \omega) \to k$ is given by some $\varphi \in \hom(\omega, \omega')$. That is, there exists a unique $\varphi : \omega \to \omega'$ with $t = t' \circ \varphi_*$. By symmetry, there exists a unique $\psi : \omega' \to \omega$ such that $t' = t \circ \psi_*$. $\psi\varphi : \omega \to \omega$ satisfies $t = t \circ (\psi\varphi)_*$, and the identity does this, and so $\psi\varphi = \mathrm{id}$ and similarly for $\varphi\psi$. □

Assume that X is projective over k, choose a closed embedding $X \subseteq \mathbb{P}^N_k$. Set $r = \operatorname{codim}(X, \mathbb{P}^N)$.

Definition 8.26. $\omega_X^* = \mathscr{E}xt_p^r(\mathscr{O}_X\omega_{\mathbb{P}^N})$.

Lemma 8.51. *$\mathscr{E}xt_P^i(\mathscr{O}_X, \omega_P) = 0$ for $i < r$.*

Proof. $\mathscr{E}xt_P^i(\mathscr{O}_X, \omega_P)$ is coherent as an $\mathscr{O}_X$-module. Thus, $\mathscr{E}xt_P^i(\mathscr{O}_X, \omega_P)(q)$ is generated by global sections for all $q >> 0$.

$\Gamma(P, \mathscr{E}xt_P^i(\mathscr{O}_X(-q), \omega_P)) \simeq H^{N-i}(P, \mathscr{O}_X(-q)) = \mathrm{Ext}_P^i(\mathscr{O}_X(-q), \omega_P)$. This is zero if $N - i > \dim X$ iff $i < r$. □

Note: Assume $0 \to M' \to M \to M'' \to 0$ ses. Then M' injective implies split exact, so $\hom(M, M') \to \hom(M', M')$ surjective. If M', M injective, then M'' injective. And $0 \to M^0 \to M^1 \to \ldots \to M^r \to 0$ with M^i injective for $i < r$, then M^r injective.

Lemma 8.52. *There is a natural isomorphism $\hom_X(-, \omega_X^\circ) \simeq \mathrm{Ext}_P^r(-, \omega_P)$ of functors from $Mod(X) \to Mod(k)$.*

Proof. $0 \to \omega_P \to \mathscr{I}^*$ an injective resolution in $Mod(P)$. Set $\mathscr{J}^m = \mathscr{H}om_P(\mathscr{O}_X, \mathscr{I}^m) \subseteq \mathscr{I}^m$ the subsheaf of sections killed by the ideal sheaf of X.

Note: $\mathscr{F}$ an $\mathscr{O}_X$-module, $\varphi : \mathscr{F} \to \mathscr{I}^m$ an $\mathscr{O}_P$-hom, then $\mathrm{Im}(\mathscr{F}) \subseteq \mathscr{J}^m$.

Thus, $\mathscr{J}^m$ is an injective $\mathscr{O}_X$-module. So $\hom_X(-, \mathscr{J}^m) = \hom_P(-, \mathscr{I}^m)$ is exact.

$H^i(\mathscr{J}^*) = \mathscr{E}xt_P^i(\mathscr{O}_X, \omega_P) = 0$ for $i < r$. So we have exact sequence $0 \to \mathscr{J}^0 \to \ldots \to \mathscr{J}^{r-1} \to \mathscr{J}^r$, and we get $\mathscr{J}^{r-1} \to \mathscr{J}_1^r \to 0$ with $\mathscr{J}_1^r = \mathrm{Im}\, d(d^{r-1}) \subseteq \mathscr{J}^r$. The Note implies that $\mathscr{J}_1^r$ is injective, so we get $0 \to \mathscr{J}_1^r \to \mathscr{J}^r \to \mathscr{J}_2^r \to 0$ with $\mathscr{J}^r = \mathscr{J}_1^r \oplus \mathscr{J}_2^r$, so all are injective.

So the exact complex can be called $\mathscr{J}_1^*$ and $\mathscr{J}_2^r \to \mathscr{J}^{r+1} \to \ldots$ can be called $\mathscr{J}_2^*$.

$\omega_X^\circ = \mathscr{E}xt_P^r(\mathscr{O}_X, \omega_P) = H^r(\mathscr{J}^*) = \ker(\mathscr{J}_2^r \to \mathscr{J}^{r+1})$, so we get $0 \to \omega_X^\circ \to \mathscr{J}_2^r \to \mathscr{J}^{r+1}$ which gives, for $\mathscr{F}$ an $\mathscr{O}_X$-module, $0 \to \hom_X(\mathscr{F}, \omega_X^\circ) \to \hom_X(\mathscr{F}, \mathscr{J}_2^r) \to \hom_X(\mathscr{F}, \mathscr{J}^{r+1})$, so $\mathrm{Ext}_P^r(\mathscr{F}, \omega_P) = H^r(\hom_P(\mathscr{F}, \mathscr{I}^*)) = H^r(\hom_X(\mathscr{F}, \mathscr{J}^*)) = H^r(\hom_X(\mathscr{F}, \mathscr{J}_2^*)) = \hom_X(\mathscr{F}, \omega_X^\circ)$. □

Proposition 8.53. *ω_X° is a dualizing sheaf for X.*

Proof. $\mathscr{F}$ is any coherent $\mathscr{O}_X$-module. $n = N - r = \dim(X)$. $\hom_X(\mathscr{F}, \omega_X^\circ) \times H^n(X, \mathscr{F}) = \mathrm{Ext}_P^r(\mathscr{F}, \omega_P) \times H^{N-r}(P, \mathscr{F}) \to H^N(P, \omega_P) \simeq k$ is a perfect pairing, by Serre Duality for $\mathbb{P}^N$.

Take $\mathscr{F} = \omega_X^\circ$. Then $\hom_X(\omega_X^\circ, \omega_X^\circ) \times H^N(X, \omega_X^\circ) \to H^N(P, \omega_P) = k$ is perfect pairing. $\mathrm{id} \in \hom_X(\omega_X^\circ, \omega_X^\circ)$ corresponds to $t : H^n(X, \omega_X^\circ) \to H^N(P, \omega_P)$, the trace map. Because everything has been natural, we can use t to factor the perfect pairing. □

Let X be a proper scheme over k of dimension n.

Definition 8.27 (Dualizing Sheaf)**.** *A dualizing sheaf for X is a coherent $\mathscr{O}_X$-module ω_X° with a k-linear trace map $t : H^n(X, \omega_X^\circ) \to k$ such that for all coherent $\mathscr{O}_X$-mod $\mathscr{F}$, we have a perfect pairing $\hom_X(\mathscr{F}, \omega_X^\circ) \times H^n(\mathscr{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{t} k$.*

Thus, it must be unique. $\omega_{\mathbb{P}^N} = \bigwedge^N \Omega_{\mathbb{P}^N}$ is the dualizing sheaf for $\mathbb{P}^N$. IF $X \subseteq \mathbb{P}^N$ is a closed subscheme of codimension r, then $\omega_X^\circ = \mathscr{E}xt^r_{\mathbb{P}^N}(\mathscr{O}_X, \omega_{\mathbb{P}^N})$ dualizing for X.

Notice: A choice of trace map t gives a natural isomorphism $\theta^\circ : \hom_X(-, \omega_X^\circ) \to H^n(X,-)^*$ of functors $Coh(X)^{op} \to Mod(k)$.

Corollary 8.54. *$\exists$ natural transformations $\theta^i : \mathrm{Ext}^i_X(-, \omega_X^\circ) \to H^{n-i}(X,-)^*$.*

Proof. We must show that both sides are δ-functors that agree for $i = 0$ and that $\mathrm{Ext}^i_X(-, \omega_X^\circ)$ is erasable in $Coh(X)^{op}$ for $i > 0$.

Choose $\mathscr{O}_X(1)$ very ample. $\mathscr{F}$ coherent implies that there exists a surjection $\mathscr{O}(-q)^{\oplus N} \to \mathscr{F}$ for $q >> 0$. $\mathrm{Ext}^i_X(\mathscr{O}(-q)^{\oplus N}, \omega_X^\circ) = \mathrm{Ext}^i_X(\mathscr{O}_X(-q), \omega_X^\circ)^{\oplus N} = \mathrm{Ext}^i(\mathscr{O}_X, \omega_X^\circ(q))^{\oplus N} = H^i(X, \omega_X^\circ(q))^{\oplus N} = 0$ for $q >> 0$, so done. □

Definition 8.28 (Regular Sequence)**.** *Let A be a local ring, $\mathfrak{m} \subseteq A$ maximal ideal, and $f_1, \dots, f_r \in \mathfrak{m}$. Then $f_1, \dots, f_r$ is a regular seqeunce if each f_i is a nonzero divisor on $A/(f_1, \dots, f_{i-1})$.*

Definition 8.29 (Cohen-Macaulay Ring)**.** *A is Cohen-Macaulay (CM) if $\exists$ a regular sequence $f_1, \dots, f_r \in \mathfrak{m}$ such that $\dim(A) = r$.*

Facts:

1. A regular local ring is CM
2. If A is CM then $f_1, \dots, f_r \in \mathfrak{m}$ is a regular sequence iff $\dim A/(f_1, \dots, f_r) = \dim A - r$.
3. A is CM and $f_1, \dots, f_r \in \mathfrak{m}$ a regular sequence, then $A/(f_1, \dots, f_r)$ is CM.
4. A CM, $f_1, \dots, f_r \in \mathfrak{m}$ a regular sequence, then $I = (f_1, \dots, f_r) \subseteq A$ implies that $(A/I)[t_1, \dots, t_r] \to \mathrm{gr}_I(A) = \oplus_{n \geq 0} I^n/I^{n+1}$ by $t_i \mapsto \overline{f_i} \in I/I^2$ is an isomorphism, so $I/I^2 \simeq (A/I)^{\oplus r}$.

Koszul Complex

A a ring, $f_1, \dots, f_r \in A$ and $A^{\oplus r}$ has basis $e_1, \dots, e_r$. Define $K_p = \bigwedge^p(A^{\oplus r})$ for $0 \leq p \leq r$. Basis $\{e_{i_1} \wedge \dots \wedge e_{i_p}\}$.

$d_1 : K_1 \to K_0$ by $e_i \mapsto f_i$. $d_p : K_p \to K_{p-1}$ by $d_p(e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum_{j=1}^p (-1)^{j-1} f_{i_j} e_{i_j} \wedge \dots \wedge \hat{e}_{i_j} \wedge \dots \wedge e_{i_p}$.

So $K_* = K_*(f_1, \dots, f_r, A)$ is this complex.

Proposition 8.55. *A a local ring, $f_1, \dots, f_r$ a regular sequence for A. Then $0 \to K_* \to A/(f_1, \dots, f_r) \to 0$ is exact.*

Definition 8.30 (Cohen-Macaulay)**.** *A scheme X is Cohen-Macaulay if $\mathscr{O}_{X,P}$ is CM for all $P \in X$.*

Definition 8.31 (Local Complete Intersection)**.** *If Y is a nonsingular variety over a field k and $X \subseteq Y$ is a closed subscheme of codim r, and $\mathscr{I}_X \subseteq \mathscr{O}_Y$ the ideal sheaf of X. Then X is a local complete intersection in Y if $\mathscr{I}_{X,P}$ is generated by r elements, $\forall P \in X$.*

Note: $X \subseteq Y$ a local complete intersection implies that X is Cohen-Macaulay and $\mathscr{I}_X/\mathscr{I}_X^2$ is a locally free $\mathscr{O}_X$-module of rank r, the "Conormal Sheaf" of X in Y.

Definition 8.32 (Normal Sheaf)**.** *$\mathscr{N}_{X/Y} = (\mathscr{I}_X/\mathscr{I}_X^2)^\vee$ is the Normal Sheaf of X in Y.*

Recall that $X \subseteq Y$ nonsingular gives $0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_Y \otimes \mathscr{O}_X \to \Omega_X \to 0$ is exact, so dualizing we get $0 \to \mathscr{T}_X \to \mathscr{T}_Y \otimes \mathscr{O}_X \to \mathscr{N}_{X/Y} \to 0$

Theorem 8.56. *$X \subseteq \mathbb{P}_k^N$ local complete intersection of codim r, then $\omega_X^\circ \simeq \omega_{\mathbb{P}^N} \otimes \bigwedge^r \mathscr{N}_{X/\mathbb{P}^n}$.*

Proof. Let $U = \operatorname{Spec}(A) \subseteq \mathbb{P}^N$ open affine such that $I(X \cap U) \subseteq A$ is generated by $f_1, \dots, f_r \in A$. Then let $P \in U \cap X$. $m = I(P) \subseteq A$ is a maximal ideal.

Then $A_m = \mathscr{O}_{\mathbb{P}^N,P}$ is regular local, so CM. $\dim A_m/(f_1, \dots, f_r) = \dim \mathscr{O}_{X,P} = \dim X = N - r$. So $f_1, \dots, f_r$ is a regular sequence.

So we look at $0 \to K_*(f_1, \dots, f_r, A_m) \to A_m/(f_1, \dots, f_r) \to 0$. Replace U with some U_h, so WLOG, $0 \to K_*(f_1, \dots, f_r; A) \to A/(f_1, \dots, f_r) \to 0$ is exact.

Thus, $0 \to \tilde{K}_* \to \mathscr{O}_{X\cap U} \to 0$ is locally free resolution on U.

Therefore, $\omega_X^\circ|_U = \mathscr{E}xt^r_{\mathbb{P}^N}(\mathscr{O}_X, \omega_{\mathbb{P}^N}) = \mathscr{E}xt^r_U(\mathscr{O}_{X\cap U}, \omega_U) = H^r(\mathscr{H}om_U(\tilde{K}_*, \omega_U))$.

$\mathscr{H}om(\tilde{K}_*, \omega_U)$ gives the sequence $\ldots \to \omega_U^{\oplus R} \to \omega_U \to 0$ by $(\sigma_1, \dots, \sigma_r) \mapsto \sum(-1)^{j-1} f_j \sigma_j$.

Thus, $\omega_X^\circ|_U \simeq \omega_U/(f_1, \dots, f_r)\omega_U = \omega_U \otimes \mathscr{O}_X$.

Note: $f_1, \dots, f_r$ form a basis of $\mathscr{I}_X/\mathscr{I}_X^2$, then $f_1 \wedge f_2 \wedge \ldots \wedge f_r$ is a basis of $\bigwedge^r(\mathscr{I}_X/\mathscr{I}_X^2)$. Thus $(f_1 \wedge \ldots \wedge f_r)^{-1}$ is a basis of $\bigwedge^r(\mathscr{N}_{X/\mathbb{P}^n})$.

We now define $\gamma : \omega_X^\circ|_U = \omega_U/(f_1, \dots, f_r)\omega_U \to \omega_U \otimes \mathscr{N}_{X/\mathbb{P}^N}$ by taking $\bar{\sigma} \mapsto \sigma \otimes (f_1 \wedge \ldots \wedge f_r)^{-1}$.

Claim: γ is independent of choices, and so defines a global isomorphism $\omega_X^\circ \to \omega_{\mathbb{P}^n} \otimes \mathscr{N}_{X/\mathbb{P}^N}$.

Let $g_1, \dots, g_r \in I(X \cap U) \subseteq A$ be another set of generators. Write $g_i = \sum c_{ij} f_j$ with $c_{ij} \in A$. Define a map $\varphi : A^{\oplus r} \to A^{\oplus r}$ by $\varphi(e_i) = \sum c_{ij} e_j$. As $\bigwedge^p$ is a functor, we get a map on exterior powers.

So we get $0 \to K_*(g_1, \dots, g_r; A) \to A/I \to 0$ and $0 \to K_*(f_1, \dots, f_r; A) \to A/I \to 0$, with vertical maps given by exterior powers of φ and the identity from A/I to itself, with everything commuting.

As $\bigwedge^r \varphi = \det(c_{ij}) : K_r(g) \to K_r(f)$ and so $\det(c_{ij}) : \mathscr{H}om_U(\tilde{K}_r(f), \omega_U) \to \mathscr{H}om_U(\tilde{K}_r(g), \omega_U)$ gives a map $\omega_U \to \omega_U$.

Tus we get map $\omega_U(f_1, \dots, f_r)\omega_U \overset{\det(c_{ij})}{\to} \omega_U/(g_1, \dots, g_r)\omega_U$ and maps from each to $\omega_U \otimes \bigwedge^r \mathscr{N}_{X/\mathbb{P}^N}$ giving a commutative diagram. □

Corollary 8.57. *If X is a nonsingular projective variety then $\omega_X^\circ = \omega_X = \bigwedge^n \Omega_X$ for $n = \dim X$.*

Proof. $X \subseteq \mathbb{P}_k^N$ closed subvariety. So $0 \to \mathscr{I}_X/\mathscr{I}_X^2 \to \Omega_{\mathbb{P}^N} \otimes \mathscr{O}_X \to \Omega_X \to 0$. Thus, $\omega_{\mathbb{P}^N} \otimes \mathscr{O}_X = \bigwedge^N(\Omega_{PP^N} \otimes \mathscr{O}_X) = \bigwedge^r \mathscr{I}_X/\mathscr{I}_X^2 \otimes \omega_X$.

Thus, $\omega_X = \omega_{\mathbb{P}^n} \otimes \bigwedge^r \mathscr{N}_{X/\mathbb{P}^N}$ □

9 Curves

Let $k = \bar{k}$ be an algebraically closed field.

Definition 9.1 (Curve)**.** *A curve is a complete connected non-singular variety of dimension 1 over k.*

Fact: All curves are projective.

$\mathrm{Div}(X) = \{D = \sum_{p\in X} m_p P\}$ and $\deg(D) = \sum m_p$.

$\mathscr{L}$ an invertible $\mathscr{O}_X$-module, $s \in \Gamma(U, \mathscr{L})$, $U \subseteq X$, then $(s) = \sum_{P\in X} v_P(s)P$ gives $\mathrm{Pic}(X) \simeq \mathrm{C}\ell(X)$ by $\mathscr{L} \mapsto (s)$, $s \in \Gamma(U, \mathscr{L})$. The inverse map takes D to $\mathscr{O}_X(D)$ by $\Gamma(U, \mathscr{O}_X(D)) = \{f \in k(X) | v_P(f) \geq -m_p\}$ where $D = \sum m_P P$

Fact: $s \in k(X)^*$, then $s \in \Gamma(U, \mathscr{O}_X)$, $\deg(s) = 0$. So $\deg : \mathrm{C}\ell(X) \to \mathbb{Z}$ is well-defined.

$\omega_X = \Omega_X$ is the canonical sheaf.

Serre Duality: $H^i(X, \mathscr{L}^{-1} \otimes \omega_X) = \mathrm{Ext}^i_X(\mathscr{L}, \omega_X) = H^{1-i}(X, \mathscr{L})^*$.

Definition 9.2 (Genus)**.** *The geometric genus is $p_g(X) = \dim_k H^0(X, \omega_X)$.*
The arithmetic genus is $p_a(X) = (-1)^{\dim X}(\chi(\mathscr{O}_X) - 1)$.

When X is a curve, $p_a(X) = \dim_k H^1(X, \mathscr{O}_X)$. By Serre Duality, we have:

Proposition 9.1. *For a curve X, $p_a(X) = p_g(X)$.*

Proof. Set $i = 0$ then Serre Duality says that $H^0(X, \mathscr{L}^{-1} \otimes \omega_X) = H^1(X, \mathscr{L})^*$. Then if $\mathscr{L} = \mathscr{O}_X$, we get $H^0(X, \omega_X) = H^1(X, \mathscr{O}_X)^*$. □

As $\omega_X \in \mathrm{Pic}(X)$, it corresponds to a divisor $K = K_X \in \mathrm{C}\ell(X)$ the Canonical Divisor, with $\omega_X = \mathscr{O}_X(K_X)$.

If $D \in \mathrm{Div}(X)$, we can define $\ell(D) = \dim_k H^0(X, \mathscr{O}_X(D))$. So, for example, $\ell(K_X) = g$, the genus of X.

Lemma 9.2.

1. *$\ell(D) \neq 0 \Rightarrow \deg(D) \geq 0$.*
2. *If $\ell(D) \neq 0$ and $\deg(D) = 0$, then $D \sim 0$ (ie, $D = 0$ in $\mathrm{C}\ell(X)$)*

Proof.

1. $\exists 0 \neq s \in \Gamma(X, \mathscr{O}_X(D))$, and $(s) = \sum_{P\in X} v_P(s)P$ and so $\deg(D) = \deg(s) \geq 0$.
2. If $\deg(D) = 0$, then $(s) = 0$, so then $D \sim 0$.

□

Theorem 9.3 (Riemann-Roch)**.** *Let $D \in \mathrm{Div}(X)$. Then*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g$$

Proof. $\ell(K - D) = \dim_k H^0(X, \mathscr{O}_X(D)^{-1} \otimes \omega_X) = \dim_k H^1(X, \mathscr{O}_X(D))$. So $\ell(D) - \ell(K - D) = \dim_k H^0(X, \mathscr{O}_X(D)) - \dim H^1(X, \mathscr{O}_X(D)) = \chi(\mathscr{O}_X(D))$.

If $D = 0$, then $\ell(0) - \ell(K) = \deg(0) + 1 - g$, that is, $1 - g = 1 - g$.

ETS that RR holds for D iff RR true for $D + P$ (P a point).

Let $k(P) = \mathscr{O}_X/\mathscr{I}_P$ and note that $\mathscr{I}_P = \mathscr{O}_X(-P)$. So we get $0 \to \mathscr{O}_X(-P) \to \mathscr{O}_X \to k(P) \to 0$ ses.

Tensor with $\mathscr{O}_X(D+P)$ and get $0 \to \mathscr{O}_X(D) \to \mathscr{O}_X(D+P) \to k(P) \to 0$ ses. As Euler Characteristic is additive, $\chi(\mathscr{O}_X(D+P)) = \chi(\mathscr{O}_X(D)) + 1$. $\square$

Examples:

1. $g - 1 = \ell(K) - \ell(K-K) = \deg(K) + 1 - g$, so $\deg(K) = 2g - 2$.
2. $\deg(D) > 2g-2$, then $\ell(K-D) = 0$, so $\ell(D) = \deg(D) + 1 - g$.
3. Assume $g(X) = 0$. Then $\deg(K) = -2$. Let $P \neq Q \in X$, $D = P - Q \in \mathrm{Div}(X)$. $\ell(D) = \ell(D) - \ell(K-D) = 0+1-0 = 1$. So then $D \sim 0$, so $P \sim Q$. Thus, $X = \mathbb{P}^1$ from last semester because $f \in k(X)$ with $(f) = P - Q$ gives a map $f : X \to \mathbb{P}^1$, which turns out to be an isomorphism.

Fact: A morphism $f : X \to Y$ of curves is finite or constant.

Definition 9.3 (Degree of a Map)**.** $\deg(f) = [k(X) : k(Y)]$.

If $P \in X$ and $Q = f(P) \in Y$ with Y a nonsingular curve, then $\mathfrak{m}_Q = (t) \subseteq \mathscr{O}_{Y,Q}$ and $f^*(t) \in \mathscr{O}_{X,P}$.

Define $e_P = v_P(f^*(t))$.

This number is called the ramification index of f at P. If $e_P = 1$, then f is unramified at P. If $e_P > 1$, then f is ramified at P. In this case, Q is a branch point.

A ramification point is called tame if $\mathrm{char}(k) = 0$ or if $\mathrm{char}(k) \not| e_P$. If it is not tame, then it is wild.

Recall: $f^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ by $f^*(Q) = \sum_{P \in f^{-1}(Q)} e_P P$. This map in fact gives a map $f^* : \mathrm{C}\ell(Y) \to \mathrm{C}\ell(X)$. This is becuase $D \in \mathrm{Div}(Y)$, then $f^*(\mathscr{O}_Y(D)) \simeq \mathscr{O}_X(f^*D)$

Definition 9.4 (Separable Map)**.** *$f : X \to Y$ is separable if it is finite and $f^* : k(Y) \subseteq k(X)$ is a separable extension of fields.*

Proposition 9.4. *$f : X \to Y$ is a separable map of curves, then $0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$ is ses.*

Proof. We already know everything except that the map $f^*\Omega_Y \to \Omega_X$ is injective. Note that these are both invertible $\mathscr{O}_X$-modules on an irreducible variety.

ETS that the map is nonzero.

Let $P_0 \in X$ be the generic point. It is enough to show that $(\Omega_{X/Y})_{P_0} = 0$. This stalk is equal to $\Omega_{k[X]/k[Y]}$ localized at 0. Taking modules of differentials commutes with localization, do we get $\Omega_{k(X)/k(Y)} = 0$ as the extension is separable. $\square$

Let $f : X \to Y$ be finite morphism of curves. $P \in X$ has image $Q = f(P) \in Y$. $\mathfrak{m}_P = (u) \subseteq \mathscr{O}_{X,P}$ and similarly $\mathfrak{m}_Q = (t) \subseteq \mathscr{O}_{Y,Q}$. Then $du \in \Omega_{X,P}$ is a generator, and similarly, $dt \in \Omega_{Y,Q}$ is a generator. $f^*(dt) \in (f^*\Omega_Y)_P \subseteq \Omega_{X,P}$. So $\exists h \in \mathscr{O}_{X,P}$ with $f^*(dt) = h du$. We will set the notation $dt/du = h$. This just means that $f^*(dt) = dt/du \cdot du \in \Omega(X,P)$.

Proposition 9.5. *Let $f : X \to Y$ be separable map of curves.*

1. $\text{supp}(\Omega_{X/Y}) = \{P \in X | e_P > 1\}$*, which is a finite set.*
2. $length_{\mathscr{O}_{X,P}}(\Omega_{X/Y,P}) = v_P(dt/du)$.
3. *If f is tamely ramified at P, then $length(\Omega_{X/Y,P}) = e_p - 1$. If f is wildly ramified, then $length(\Omega_{X/Y,P}) > e_p - 1$.*

Proof. 1. $\Omega_{X/Y,P} = 0$ iff $\Omega_{X,P}$ generated by $f^*(dt)$ iff f^*t generates $\mathfrak{m}_P \subseteq \mathscr{O}_{X,P}$ iff $e_P = 1$.

2. $(\Omega_{X/Y})_P \simeq \Omega(X,P)/f^*\Omega_{Y,Q} = \Omega_{X,P}/(f^*(dt)) \simeq \mathscr{O}_{X,P}/(dt/du)$.

3. Set $e = e_P$ and $t = f^*(t)$. Then $t = au^e$ with $a \in \mathscr{O}_{X,P}$ a unit. Then $dt = dau^e + aeu^{e-1}du$. If the ramification is tame, then $e \neq 0 \in \mathscr{O}_{X,P}$, so, as $da = hdu$ for some h, $v_P(dt/du) = e - 1$.

   If wild, then $e = 0 \in \mathscr{O}_{X,P}$, so $dt = u^e da$, so $v_P(dt/du) = v_P(u^e h) >\geq e > e - 1$.

∎

Assume from now: $f : X \to Y$ separable finite morphism of curves.

Definition 9.5 (Ramification Divisor)**.** $R = \sum_{P \in X} length(\Omega_{X/Y,P})P \in Div(X)$.

Proposition 9.6. $K_X \sim f^*K_Y + R$.

Proof. $s \in \Gamma(V, \Omega_Y)$, then $K_Y = (s) \in \text{Div}(Y)$. $f^*(s) \in \Gamma(f^{-1}(V), f^*\Omega_Y) \subseteq \Gamma(f^{-1}(V), \Omega_X)$. $K_X = (f^*s) \in \text{Div}(X)$.

Let $(K_X; P)$ denote the coefficient of P in K_X. Then $(K_X; P) = length_{\mathscr{O}_{X,P}}(\Omega_{X,P}/(f^*s)) = length(f^*\Omega_{Y,P}/(f^*s)) + length(\Omega_{X/Y,P}) = (f^*K_Y; P) + (R; P)$. ∎

Theorem 9.7 (Hurwitz Theorem)**.** $2g(X) - 2 = n(2g(Y) - 2) + \deg(R)$ *where n is* $\deg(f)$.

Example: $f : X \to Y$ any finite morphism of curves, then $g(X) \geq g(Y)$. Why? Because $g(X) = g(Y) + (n-1)(g(Y) - 1) + \frac{1}{2}\deg(R)$.

The ramification divisor has even degree.

Theorem 9.8 (Lyroth's Theorem)**.** *If $k \subseteq L \subseteq k(t)$ and L is a field, then $k = L$ or $L = k(u)$ for some $u \in L$.*

Proof. $k \neq L$, then k has transcendence degree 1, so let $X = C_L$. Then $L \to k(t)$ gives a map $f : \mathbb{P}^1 \to X$ a finite map of curves. As $g(\mathbb{P}^1) \geq g(X)$ and $g(\mathbb{P}^1) = 0$, $g(X) = 0$, so $X \simeq \mathbb{P}^1$. Thus, $L \simeq k(u)$ for some $u \in L$. ∎