# Lecture 1

Algebra of the Complex Numbers

Topology of Complex Numbers.

Basics.

Definition of continuous.

**Definition 1.1 (Holomorphic).** A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at $z \in \mathbb{C}$ if $\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$ exists. We denote it by $f'(z)$.

If $f$ has holomorphic derivative at $z_0$, then $f(z_0 + h) = f(z_0) + ah + h\psi(h)$ where $\lim_{h \to 0} \psi(h) = 0$.

The usual rules of differentiability: linearity, leibniz, quotient, and chain all hold.

Cauchy Riemann Equations $u_y = -v_x$ and $u_x = v_y$.

# Lecture 2

We are discussing equivalent notions of "holomorphic."

1. Holomorphic: $f$ is a $\mathbb{C}$-valued function defined on $\Omega \subseteq \mathbb{C}$ such that $\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = f'(z)$ exists for all $z \in \Omega$.

2. Cauchy-Riemann Equations: assume that $u$ and $v$ have continuous partials, then $u_x = v_y$ and $u_y = -v_x$. That is, $\partial_{\bar{z}}(u + iv) = 0$.

3. Conformal map: $f$ preserves angles infinitesimally. This is true if the Jacobian of $f$ as a map $\mathbb{R}^2 \to \mathbb{R}^2$ is a constant times a rotation matrix. This is equivalent to C-R.

A power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then if $1/R = \limsup_{n \to \infty} |a_n|^{1/n}$, then the power series converges absolutely and uniformly on any disc $D_r(0)$ with $r < R$ and it diverges at any $z$ with $|z| > R$.

Proving basic theorems about power series.

A function with a convergent complex power series $\sum a_n(z - z_0)^n$ is called an analytic function.

Analytic implies Holomorphic in the disc of convergence.

If $\phi$ is $C^\infty$, then $|\phi - \sum_{j+k \leq N} a_{jk} z^j \bar{z}^k|$ is $O(|z|^{N+1})$. This is generally not convergent. If $\partial_{\bar{z}} f = 0$, then $a_{jk} = 0$ for $k \geq 1$.

For an analytic function, we have $a_{jk} = \frac{\partial^k \phi}{k!}(0)$.

**Proposition 2.1.** If $f$ is analytic in $D_r(0)$, then the power series for $f$ about $z_0 \in D_R(0)$ converges in $D_{R-|z_0|}(z_0)$.

Basic definitions for contour integration.

**Theorem 2.2.** If $D \subset \mathbb{C}$ is a bounded region with piecewise smooth boundary and $\omega$ is a smooth $C^1$ 1-form, then $\oint_{\partial D} f dz = \int_D d(fdz) = \int_D \partial_z f dz \wedge dz$.  

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So the easy form of Cauchy’s theorem is that if $f$ is holomorphic on $\Omega$ and $D \subset \subset \Omega$, is bounded with piecewise $C^1$ boundary and $f \in C^1(\Omega)$, then $\int_{\partial D} f \, dz = 0$.

**Theorem 2.3** (Goursat). Let $f$ be holomorphic in a domain $\Omega$ and $T$ a triangle such that $D_T$, the region with $bD_T = T$ contained in $\Omega$. Then $\int_{T} f(z) \, dz = 0$.

See Stein-Shakarchi for all this technical babble.

### 3 Lecture 3

**Theorem 3.1.** If $f$ is holomorphic in a disk $D_R(z_0)$, then $f$ has a primitive in $D_R(z_0)$.

**Corollary 3.2.** If $f$ is holomorphic on $D_R(z_0)$ then $\int_{\gamma} f \, dz = 0$ for any closed piecewise $C^1$ curve $\gamma \subset D_R(z_0)$.

**Theorem 3.3.** Assume that $f$ is holomorphic on a neighborhood of $D_R$ where $bD_R$ is a rectangle along with a single point $z_0 \in D_R$ and there exist an $A > 0$ and an $0 \leq \alpha < 1$ such that $|f(z)| \leq A/|z-z_0|^{\alpha}$ near to $z_0$. Then $\int_{D_R} f(z) \, dz = 0$.

Derive the Cauchy integral formula

Now that we know that $f$ has continuous derivatives, we can apply Stoke’s Theorem.

Holomorphic implies $C^\infty$.

**Theorem 3.4** (Cauchy Estimates). If $f$ is holomorphic in the disc of radius $r$ centered at $z_0$ and continuous up to the boundary with $|f(z)| \leq M$ for $|z-z_0| = r$, then $|f^{(n)}(z_0)| \leq Mn!/r^n$.

**Theorem 3.5.** If $f$ is holomorphic in a set containing $D_R(z_0)$ then the Taylor series of $f$ about $z_0$ converges on $D_R(z_0)$ is $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$.

That is, holomorphic iff analytic.

### 4 Lecture 4

We will look at $\partial \Omega f = \varphi$. $f(z) = \frac{1}{2\pi i} \int_{bD_R(z_0)} \frac{f(w) \, dw}{w-z}$ when $f$ is holomorphic on $D_r(z_0)$ and continuous up to the boundary. Sometimes we will assume that $f$ is holomorphic on $D_r(z_0)$, that is, on some open set containing this.

**Definition 4.1** (Entire). A function holomorphic at every point of $\mathbb{C}$ is called entire.

**Theorem 4.1** (Liouville’s Theorem). A bounded entire function is constant.

**Proof.** $f'(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w) \, dw}{w-z}$. $\left|f(w)\right| \leq M$ for all $w \in \mathbb{C}$. Then $|f'(z)| \leq \frac{1}{2\pi} \int_{|w|=R} \frac{|f(w)| \, dw}{|w-z|^2}$. As $|w-z| \geq |w| - |z|$, we have $\leq \frac{1}{2\pi} \int_{|w|=R} \frac{M \, dw}{(R-|z|)^2} \leq \frac{M R}{(R-|z|)^2}$. Taking $R \to \infty$, $f'(z) = 0$. $\square$

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Corollary 4.2 (Fundamental Theorem of Algebra). Let $p(z)$ be a polynomial
\[ \sum_{j=0}^{n} a_j z^j \] with $a_n \neq 0$ and $n > 0$. Then there exists $z_0 \in \mathbb{C}$ where $p(z_0) = 0$.

**Proof.** Assume not, then $\frac{1}{p(z)}$ is entire. \[ |p(z)| \geq |a_n z^n| - \sum_{j=0}^{n-1} |a_j z^j| = a_n |z|^n \left(1 - \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|} \frac{|z|^j}{|z|^n-j} \right). \] Thus, there exists $R$ such that $|p(z)| \geq \frac{|a_n z^n|}{2}$ for $|z| \geq R$. Thus, $1/p(z)$ is bounded, and so constant by Liouville. Thus, $p$ was constant, which is a contradiction.

To get the fact that $p$ has $n$ roots, let $z_1$ be a root of $p$. Then $p = \sum_{j=1}^{n} p_{j}(z_1) / j!(z - z_1)^j$. We can write this as $\sum_{j=0}^{n-1} p_{j+1}(z_1) / (j + 1)(z - z_1)^j$. Repeating, we get this fact.

Suppose that $f_n$ is a sequence of $C^1$ functions. Then we need to know that $f'_n$ is uniformly convergent to know that $\lim_{n \to \infty} f_n$ is $C^1$.

**Theorem 4.3.** If $f_n$ is a sequence of holomorphic functions on $\Omega \subset \mathbb{C}$ which is locally uniformly convergent to $f$, then $f$ is holomorphic.

**Lemma 4.4.** If $\varphi$ is a continuous function on $bD_r(z_0)$ then $F(w) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{\varphi(w)}{w-z} \, dw$ is holomorphic in $\mathbb{C} \setminus bD_r(z_0)$.

We now prove the theorem

**Proof.** If $K \subset \subset \Omega$ then $f_n \to f$ uniformly on $K$. \[ \|f_n - f\|_{L^\infty(K)} \to 0. \]

If $z \in \Omega$, then there exists $r > 0$ such that $D_r(z) \subset \subset \Omega$. For $\zeta \in D_r(z)$, we can write $f_n(\zeta) = \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f_n(w)\, dw}{w-\zeta}$ for all $n$. Thus $f(\zeta) = \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f(w)\, dw}{w-\zeta}$.

This same argument on the derivatives proves more.

**Theorem 4.5** (Bonel’s Lemma). Given a sequence of numbers $\{a_j\}$, there exists a $C^\infty$ function $\phi$ such that $\phi^{(j)}(0) = a_j$ for all $j$.

However, for holomorphic functions, we have bounds given by the Cauchy Inequalities.

**Lemma 4.6.** If $f$ is holomorphic and does not vanish in some neighborhood of $z_0$, then there exists $k$ such that for $j < k$ we have $f^{(j)}(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$.

**Theorem 4.7.** If $f$ is holomorphic in $\Omega$, then $Z_f = \{z \in \Omega | f(z) = 0\}$ is discrete.

**Proof.** Let $z_n \in Z_f$ have a limit point in $\Omega$, call it $z^*$. Then $f(z^*) = 0$...

**Corollary 4.8.** If $f, g$ are holomorphic in $\Omega$ a connected domain and $f(z_n) = g(z_n)$ on a sequence of distinct points with a limit in $\Omega$, then $f \equiv g$.

**Lemma 4.9.** If $f$ is a holomorphic function on connected $\Omega$ such that $f(z) = 0$ for $z \in U \subseteq \Omega$ an open set, then $f \equiv 0$.

Stated Morera’s Theorem
5 Lecture 5

Let $f_n$ be holomorphic in $\Omega$ and $f_n \to f$ locally uniformly. Then $\int_T f(z) dz = \lim_{n \to \infty} \int_T f_n(z) dz = 0$ when $T$ is a triangle which is contained in $\Omega$. Thus, Morera’s Theorem implies $f$ holomorphic.

A complex measure $d\mu$ is a measure for the form $d\mu_1 + id\mu_2$ for $d\mu_1, d\mu_2$ real measures. Define $|d\mu_j| = d\mu_j^+ + d\mu_j^-$, $d\mu_j^+ \perp d\mu_j^-$, and it is of finite total variation $\int C |d\mu| < \infty$.

**Theorem 5.1.** Let $d\mu$ be a complex measure of finite total variation and compact support on $\mathbb{C}$. $\int_C d\mu = \int_K d\mu$ for all $f \in C^0(\mathbb{C})$ and $K$ is some compact subset of $\mathbb{C}$. We define $f(z) = \int_K \frac{d\mu(w)}{w-z}$ is defined for $z \in \mathbb{C} \setminus K$.

Then $f$ is holomorphic in $\mathbb{C} \setminus K$.

**Proof.** This is a local statement, so choose a point $z \in \mathbb{C} \setminus K$ and $r > 0$ such that $D_r(0) \subset \mathbb{C} \setminus K$.

If $T$ is a triangle contained in $D_r(z)$, then $\int_T f(z) dz = \int_T \left( \int_K \frac{d\mu(w)}{w-z} \right) dz = \int_T \int_K \left| \frac{d\mu(w)}{|w-z|} \right| |dz| < \frac{|\mu(K)|}{r} < \infty$.

So we can interchange the order of integration. Thus, $\int_K \left( \int_T \frac{dz}{w-z} \right) d\mu(z)$ for $z \in D_r(z_0)$, we have $z \mapsto \frac{1}{w-z}$ is holomorphic, so this integral is zero. Thus, Morera’s Theorem shows that $f(z)$ is analytic in $\mathbb{C} \setminus K$.

If $d\mu$ is $\phi(w)dw$ for some $\phi \in C^0(\gamma)$ along a curve $\gamma$, then $\int \psi d\mu = \int_\gamma \psi \phi dw$ and $|d\mu| = |\phi| |ds|$.

So $\int_\gamma \frac{\phi(w)dw}{w-z}$ is holomorphic in $\mathbb{C} \setminus \gamma$.

If $\phi \in C^\infty(\mathbb{C})$, then $f(z, \bar{z}) = \frac{1}{2\pi i} \int \frac{\phi(w, \bar{w})dw\wedge d\bar{w}}{w-z}$ becomes $-\frac{1}{\pi} \int \frac{\phi(a-t,b-s)dsdt}{t+is}$, where $z = a + ib$.

The key is that $\frac{1}{z}$ is an integrable singularity in the plane.

$f$ has the same number of continuous derivatives as $\phi$.

Look at $\partial_j f = \psi$, $\frac{1}{b} \int_{a,b} f_h' - f_a' = \frac{1}{x} \int f_h'(a-h,b-s) - f_a(a-h,b-s) dsdt$, so then $\lim_{h \to 0} \frac{1}{x} \int f_h'(a-h,b-s) dsdt$.

and so $\partial_j \partial_j f(a, b) = \frac{1}{x} \int \frac{\partial_j^2 \phi(a-t,b-s)}{s+it} dsdt$.

So then $\partial_z f = \frac{1}{2\pi i} \int \frac{\partial \phi(w,z)dw\wedge d\bar{w}}{w-z} = \frac{1}{2\pi i} \int \frac{\partial \phi(w,z)dw\wedge d\bar{w}}{w-z} = \lim_{c \to 0} -\frac{1}{2\pi i} \int |w|>c d \left( \frac{\phi(w+z)dw}{w} \right) = -\lim_{c \to 0} -\frac{1}{2\pi i} \int |w|=c d \left( \frac{\phi(w+z)dw}{w} \right)$ in the clockwise direction, taking the counterclockwise, we obtain $\lim_{c \to 0} \frac{1}{2\pi i} \int \frac{\phi(z+ce^{it})ce^{it}d\bar{w}}{ce^{it}} = \phi(z)$.

Thus $d \frac{1}{2\pi i} \int \frac{dw}{w-z} = \delta_0$.

**Theorem 5.2.** If $\phi \in C_1^1(\mathbb{C})$, then the equation $\partial \phi = \psi$ has a solution $f \in C^1(\mathbb{C})$ given by $f = \frac{1}{2\pi i} \int \frac{\phi(w+z)dw}{w-z}$.
Theorem 5.3 (Cauchy-Pompeiu Formula). If \( \phi \) is a \( C^1 \) function on \( \overline{D_r(z_0)} \), then

\[
\phi(z, \bar{z}) = \frac{1}{2\pi i} \left( \int_{\partial D_r(z_0)} \frac{\phi \, dw}{w - z} + \int_{D_r(z_0)} \frac{\partial_r \phi \, dw \wedge d\bar{w}}{w - z} \right)
\]

Let \( f = u + iv \), then \( u_x = v_y, \ u_y = -v_x, \ v_{yy} + u_{xx} = 0 \), and \( u_{xx} + u_{yy} = 0 \). Thus, \( \Delta u = 0 \).

Now let \( u \) such that \( \Delta u = 0 \).

Define \( dv = v_x \, dx + v_y \, dy \) and \( d^*u = -u_y \, dx + u_x \, dy \), and so \( dv = d^*u \).

Define \( d(adx + bdy) = (b_x - a_y) \, dx \wedge dy \), and so we need \( d^*u \) to be \( d \)-closed, then \( d(d^*u) = (u_{xx} + u_{yy}) \, dx \wedge dy \), and so \( \Delta u = 0 \) if and only if \( d(d^*u) = 0 \).

I can define \( \phi \) such that \( \Delta \phi = 4\pi \). Now let \( \phi(0) \) and \( \phi \) is harmonic in \( \Omega \) simply connected and \( \Delta \phi = 0 \), then there exists a function \( v \) defined on \( \Omega \) such that \( dv = d^*u \), which is equivalent to \( u + iv \) is holomorphic.

If we have two such solutions, then \( d(v_1 - v_2) = 0 \), and so \( v \) is determined up to a constant.

\( v \) is called the conjugate harmonic function.

If \( u \) is harmonic on \( D_R(0) \) and continuous to \( bD_R(0) \), then \( u(z, \bar{z}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - |z|^2)u(re^{i\theta}, re^{-i\theta})}{|Re^{i\theta} - z|^2} \, d\theta \)

for \( z \in D_r(0) \).

\[
\Delta = 4\partial_r \partial_{\bar{z}} = 4\partial_r \partial_{\bar{z}}
\]

is why we care about this representation of the kernel.

Looking at \( \int u(e^{i\theta}) P(e^{i\theta}, z) \, d\theta \), we see

1. \( P(e^{i\theta}, z) > 0 \)
2. \( \int_0^{2\pi} P(e^{i\theta}, z) \, d\theta = 1 \)
3. \( \forall \delta > 0, \ \epsilon > 0 \), there exists \( r_{\delta, \epsilon} \) such that \( \int_{|e^{i\theta} - e^{i\theta}| < \delta} P(e^{i\theta}, re^{i\theta}) \, d\theta < \epsilon \) if \( r > r_{\delta, \epsilon} \).

\[
\lim_{r \to R} u(re^{i\theta}, re^{-i\theta}) = \phi(RE^{i\theta}, Re^{-i\theta}). \text{ Then } u(z, \bar{z}) = \int \phi(P(e^{i\theta}, z) \, d\theta, \quad |u(re^{i\theta}, re^{-i\theta}) - \phi(e^{i\theta})| = \int |\phi(e^{i\theta}) - \phi(e^{i\theta})| P(e^{i\theta}, re^{i\theta}) \, d\theta \leq \int |\phi(e^{i\theta}) - \phi(e^{i\theta})| P(e^{i\theta}, re^{i\theta}) \, d\theta.
\]

\( \phi \) is continuous on the boundary, there exists \( \delta \) such that \( |\phi(e^{i\theta}) - \phi(e^{i\theta})| < \epsilon \) if \( e^{i\theta} - e^{i\theta} < \delta \), because it is a continuous function on a compact set, and thus uniformly continuous.

So \( \int_{|e^{i\theta} - e^{i\theta}| < \delta} |\phi(e^{i\theta}) - \phi(e^{i\theta})| P(e^{i\theta}, re^{i\theta}) \, d\theta \leq 2 \int_{|e^{i\theta} - e^{i\theta}| < \delta} 2M P(e^{i\theta}, re^{i\theta}) \, d\theta \). Now, there exists \( r_{\delta, \epsilon} \) such that if \( r > r_{\delta, \epsilon} \), then the second term is less than \( \epsilon \), and so \( |u(re^{i\theta}) - \phi(e^{i\theta})| < 2\epsilon \).

This is in fact uniform in the boundary point.

So \( u(z, \bar{z}) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}, \bar{z} + re^{-i\theta}) \, d\theta \). This is the mean value property of harmonic functions.
6 Lecture 6

Theorem 6.1. Let $F(z, s)$ be defined in $\Omega \times [0, 1]$ with $\Omega \subset \mathbb{C}$ such that $F$ is continuous and $F$ is holomorphic in $z$.

Then $f(z) = \int_0^1 F(z, s)ds$ is holomorphic in $\Omega$.

Proof. By Morera's Theorem, it suffices to show that $\int _T f(z)dz = 0$ for all triangles $T \subset \Omega$ such that $bD_T = T$ where $\partial_D T \subset \Omega$. Since $F$ is continuous in $\Omega \times [0, 1]$, it is continuous on $T \times [0, 1]$. Since this is a compact subset of $\Omega \times [0, 1]$, $|F(z, s)| \leq M$ on $T \times [0, 1]$.

$$\int_T f(z)dz = \int_0^1 \int_C F(z, s)dzds = \int_0^1 0ds = 0.$$ \hfill \Box

Look at $\Gamma(z) = \int_0^\infty e^{-t}t^{-1}dt$. Then define $\Gamma_n(z) = \int_{1/n}^n e^{-t}t^{-1}dt$. Then the theorem applies to show that $\Gamma_n(z)$ is an entire function.

Claim: $\Gamma_n(t)$ is locally uniformly convergent in $Re(z) > 0$ to $\Gamma(z)$. $|t^z| = t^{Re(z)}$. If $u > m$, then $\Gamma(u(z)) - \Gamma_m(z) = \int_{1/m}^1 e^{-t}t^{-1}dt + \int_m^u e^{-t}t^{-1}dt$.

$$|\Gamma(u(z)) - \Gamma_m(z)| \leq \int_{1/m}^1 e^{-t}Re(z)^{-1}dt + \int_m^u e^{-t}Re(z)^{-1}dt \leq \int_{1/m}^1 e^{-t}dt + \int_m^u e^{-t}dt \leq \frac{1}{e} \left( \frac{1}{m} \right)^\delta + 2e^{-m}M.$$ So then given $\epsilon > 0$, there exists $m_0$ with $m > m_0$ implying that $|\Gamma_n(z) - \Gamma_m(z)| < \epsilon$ on $\delta \leq Re(z) \leq M$.

Now, if $f$ is holomorphic on $D_r(z_0)$, then Cauchy’s formula implies that $f(z_0) = \frac{1}{2\pi i} \int_{\partial_D} f(z + re^{i\theta})d\theta$.

$$|f(z_0)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})|d\theta.$$ FREQUENTLY WE NEED THAT $f(z_0 + re^{i\theta}) = g_r(\theta)e^{i\phi}$. Suppose that $|f(z_0)|$ assumes a local maximum at $z_0$. Then $|f(z_0 + re^{i\theta})| \leq |f(z_0)| = M$ if $r < \rho$.

$$M \leq \frac{1}{2\pi} \int_0^{2\pi} M|d\theta|.$$ THIS CAN ONLY HAPPEN IF $g_r(\theta) = M$ FOR $r < \rho$. THEN $f(z_0 + re^{i\theta}) = M e^{i\phi}$ FOR $r < \rho$.

Theorem 6.2 (Maximum Modulus Principle). If $f$ is holomorphic in $\Omega$ a connected open set and $|f(z)|$ has an interior local maximum, then $f$ is constant on $\Omega$. If $D \subset \subset \Omega$, then $|f(z)|$ assumes its maximum on $bD$ and if it assumes this value in $D$, then $f$ is constant.

6.1 Harmonic Functions

A $C^2$ function $u$ defined on $\Omega \subset \mathbb{C}$ is harmonic if $\Delta u = 0$. If $u$ is harmonic on $D_r(0)$, then for $z \in D_r(0)$, $u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})r^2 - |z|^2 \frac{d\theta}{r^2}$.

A consequence of $\Delta u = 0$ is the mean value property for harmonic functions,

$$\Delta(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})d\theta.$$ THERE EXISTS $v$ SUCH THAT $u + iv$ IS HARMONIC AND THEREFORE $(u + iv)(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) + iv(z_0 + re^{i\theta})d\theta$. TAKING REAL PARTS, WE GET THE RESULT.

Theorem 6.3 (Maximum Principle). If $u$ is harmonic in $\Omega$ a connected open set, then $u$ has no interior local maxima or minima unless $u$ is constant.
Proof. Suppose that \( u(x, y) \) has a local maximum. \( u(x, y) \leq u(x_0, y_0) \) on a small disc around \( (x_0, y_0) \) in \( \Omega \). MVP implies that \( u(x, y) = u(x_0, y_0) \) in this disc. The set where \( u(x, y) = \max_{(x, y) \in \Omega} u(x, y) \) is both open and closed. Either \( u(x, y) < \max_{\Omega} u \) or it is constant.

The same argument for minimum.

Uniqueness of the solution to Dirichlet Problem:

**Dirichlet Problem**: Let \( \Omega \subset \mathbb{C} \) be a connected bounded set with piecewise \( C^1 \)-boundary. Let \( \phi \in C^0(\partial \Omega) \). Find a harmonic function \( u \in C^2(\Omega) \cap C^0(\bar{\Omega}) \) such that \( u|_{\partial \Omega} = \phi \).

Let \( \mathcal{F}_\phi = \{ u \in C^2(\Omega) \cap C^0(\bar{\Omega}) | u|_{\partial \Omega} = \phi \} \). Define \( D(u) = \int_\Omega |\nabla u|^2 \).

Then the solution to Dirichlet’s problem is the function \( u_0 \) such that \( D(u_0) = \min_{u \in \mathcal{F}_\phi} D(u) \).

Let \( \int \nabla u_0 \nabla \psi = 0 \) for all \( \psi \in C^\infty_c(\Omega) \). Thus, \( -\int \Delta u_0 \psi = 0 = \Delta u_0 = 0 \).

Problems: Why should minimum exist? And there exists \( \phi \) such that all the Dirichlet integrals are infinite.

Uniqueness for the Dirichlet Problem: Let \( u \) be harmonic on \( \Omega \) and continuous on \( \bar{\Omega} \) with \( u|_{\partial \Omega} = 0 \). Then \( u \equiv 0 \). The maximum principle forces it.

**Theorem 6.4** (Schwarz Reflection Principle). Let \( f \) be holomorphic on \( D_r^+(0) \). Assume that \( f \) is continuous down to the real axis and that the imaginary part of \( f \) along the real line is zero. Then letting \( F(z) = \begin{cases} f(z) & \text{Im}(z) \geq 0 \\ \overline{f(z)} & z \in D_r^-(0) \end{cases} \) defines an analytic continuation of \( f \) to \( D_r(0) \).

**Proof.** \( F \) is certainly continuous in \( D_r(0) \) and \( \sum_{j=0}^\infty a_j(z-z_0)^j = \sum_{j=0}^\infty \overline{a_j}(z-z_0)^j \). So now we take \( \int_T F(z)dz = 0 \) for all \( T \subset D_r(0) \). Thus, Morera implies that \( F \) is holo.

But we actually get a stronger statement if we look at what the hypotheses really are.

1. \( \text{Re}(f) \) is continuous up to \( D_r(0) \cap \mathbb{R} \).
2. \( \text{Im}(f) \) is cont and vanishes along \( D_r(0) \cap \mathbb{R} \).

The first is unnecessary.

In fact, the theorem remains true without 1, and there is a reflection principle for harmonic functions:

**Theorem 6.5.** Let \( v \) be a real valued harmonic function on \( D_r^+(0) \) with \( \lim_{y \to 0^+} u(x, y) = 0 \). Then define \( V(x, y) = \begin{cases} u(x, y) & y \geq 0 \\ -u(x, -y) & y < 0 \end{cases} \) then \( V \) is harmonic in \( D_r(0) \), in particular, it is smooth.

Let \( \phi(x, y) \) be this function. then define \( V(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \phi(re^{i\theta}) P(e^{i\theta}, (x, y))d\theta \).
Look at \( \int v(r e^{i \theta}) \frac{r^2 - |z|^2}{|z - re^{i \theta}|^2} \) Now we note that \( P(e^{i \theta}, x) = P(e^{-i \theta}, x) \), and so \( \phi(e^{i \theta}) = -\phi(e^{-i \theta}) \), so \( V(x, 0) = 0 \).

And so \( V \) is harmonic on \( D_t(0) \), and \( V|_{bD_t^+(0)} = v|_{bD_t^+(0)} \). By uniqueness of the solution to the dirichlet problem, we have that \( V = v \) for all of \( D_t^+(0) \).

Let \( -U \) be the conjugate function to \( V \). Then \( dU = -d^*V \). So then \( du = -d^*v \) on \( D_t^+(0) \) and \( d(U - u) = 0 \) on \( D_t^+(0) \).

We can normalize such that \( U = u \) on \( D_t^+(0) \) and then \( U + iV \) is holomorphic. And so we have Schwarz without the condition on the real part.

**Theorem 6.6** (Runge). Suppose that \( f \) is holomorphic in \( D_R(0) \). Then for all \( r < R \) and \( \epsilon > 0 \) there exists a polynomial \( p(z) \) such that \( \|f - p\|_{L^\infty(D_R(0))} < \epsilon \).

This version of Runge’s Theorem is an immediate corollary of Taylor’s Theorem.

7 Lecture 7

Look at \( \partial_z u = \phi \) for any \( \phi \in C^\infty(\mathbb{R}^2) \). If \( \phi \in C_c^\infty(\mathbb{R}^2) \) then \( u = \frac{1}{2 \pi i} \int \frac{\phi(w) d\bar{w} \wedge dw}{w - z} \).

The method is called the oil spot method. Choose a sequence of functions \( \psi_j \in C_c^\infty(\mathbb{R}^2) \) such that \( \psi_j(z) \) is one for \( |z| \leq j + 1/2 \) and zero for \( |z| > j + 2 \).

Let \( u_1 \) be the solution above to \( \partial_z u_1 = \psi_1 \phi \). Let \( u_2 \) solve \( \partial_z u_2 = \psi_2 \phi \).

Then \( \partial_z (u_2 - u_1) = 0 \) on \( D_{j+1/2}(0) \). So now I can choose a holomorphic polynomial \( p_1(z) \) such that \( \|u_2 - u_1 - p_1\|_{L^\infty} < 1/2^j \). Let \( u_2 = u_2 - p_1 \). Then \( u_2 \) solves the equation as well. Define higher versions similarly.

It is obvious that \( \lim_{n \to \infty} u_n(z) = u(z) \) exists for all \( z \in \mathbb{C} \). But it is really uniform. \( \sum_{n=1}^\infty |u_{n+1} - u_n| < \infty \).

So now \( u(z) \) is a continuous function. Fix an \( R \) and choose \( n_0 > R \). Then \( u_n(z) - u_m(z) + u_m(z) = u_n(z) \). Take the first difference and call that \( h_n(z) \). Then \( h_n(z) \to h(z) \).

\( \partial_z u_n_m = \phi_m \) for \( |z| < R \) and \( n > n_0 \). So for \( n > n_0 \), \( h_n(z) \) is holomorphic in \( |z| < R \).

So \( h_n \) is a uniformly convergent sequence of holomorphic functions of \( m \) in \( |z| < R \) with limit \( h(z) \). In \( |z| < R - 1 \), \( h_n \to h \) in the \( C^\infty \) topology. \( \|\partial_z^j \partial_{\bar{z}}^k (h_n - h)\|_{L^\infty} \to 0 \) as \( n \to \infty \) for all \( j, k \).

Hence, \( u_n \to u \) locally uniformly in the \( C^\infty \) topology, and therefore \( \partial_z u = \phi \) as well on all of \( \mathbb{C} \).

This is related to \( H^1(\mathbb{C}, \partial) = 0 \).

That was a consequence of the most trivial version of the Runge Theorem.

**Theorem 7.1** (Runge). Let \( K \subset \mathbb{C} \) be a compact subset and suppose that \( f \) is holomorphic on \( U \) an open set containing \( K \). For any \( \epsilon > 0 \), there exists a function \( F_\epsilon \) of the form \( F_\epsilon(z) = \sum_{j=0}^N \sum_{\ell=0}^m \frac{a_{j,\ell}}{(z - z_j)} + p(z) \) where \( \{z_j\} \in K^C \), \( \|f - F_\epsilon\|_{L^\infty} < \epsilon \) and if \( K^C \) is connected, then \( F_\epsilon \) can be taken to be a polynomial.
Proof. Cover $\mathbb{C}$ with a grid of size $\delta$ where $\delta \leq \frac{1}{2\pi} d(K, U^c)$. Take $\{Q_j\}$ to be the list of squares that intersect $K$. Then $Q_j \subset U$ for all $j$, by the choice of $\delta$. Choose a point $z \in K$ and consider $\sum_{j=1}^{N} \frac{1}{2\pi} \int_{\partial Q_j} \frac{f(w)}{w-z} dw = f(z)$ for $z \not\in \partial Q_j$.

$$ f(z) = \sum_{j=1}^{M} \frac{1}{2\pi} \int_{\partial Q_j} \frac{f(w)}{w-z} dw $$

for all $z \in K$. $\gamma_j \cap K$ is empty for all $j$. Let $\inf_j \text{dist}(\gamma_j, K) = d > 0$.

Because $\int |w-z| \geq d$, $|f(z) - \sum_{j=1}^{n} \frac{f(w_j)}{w_j - z} m_j| < \epsilon$ for $z \in K$.

So then $w \mapsto f(w)/(w-z)$ is uniformly equicontinuous as a family of functions parameterized by $z$ and uniformly bounded. Thus, we can find a single partition of $\cup \gamma_j$ such that this gives a uniformly accurate approximation to the integrals for all $z \in K$.

$$ \{w_j\} \subset K^c \land \text{so these are functions on the desired set.} $$

$$ f_{w_{j+1}} = f_{w_j} - \int_{w_j}^{w_{j+1}} \frac{f(w)}{w-z} dw. $$

Suppose that $K \subseteq D_{R}(0)$ and $w_j \in D_{R+\epsilon}(0)^c$. Then $1/(w_j - z) = \frac{1}{w_j} \frac{1}{1 - \frac{z}{w_j}} = \frac{1}{w_j} \sum (z/w_j)^n$. Let $z \in K$ then $|z/w_j| < R/(R + \epsilon)$.

Thus, $\left| \frac{1}{w_j} - \sum (z/w_j)^n \right| < \eta$ and so $\left| \sum (z/w_j)^{n+1} \right| < \eta$.

For each $w_j$ in the unbounded component of $K^c$, there is a $C^1$ curve $\gamma_j(t)$ such that $\gamma_j(0) = w_j$, $\gamma_j(1) = w_j \in D_{R+\epsilon}(0)^c$ and $\int_{\gamma_j} d(\gamma_j, K) = \rho > 0$.

Lemma 7.2. Suppose that $w, w' \in \gamma_j([0,1])$ with $|w-w'| < \rho/2$, then given $\eta > 0$, there exists an $N$ such that $\left| \frac{1}{w-w'} - \sum_{j=1}^{N} \frac{a_j}{(w-w')^j} \right| < \eta$ for all $z \in K$.

Proof. $\frac{1}{z-w} = \frac{1}{z-w'} - \frac{1}{z-w'} \sum_{j=0}^{\infty} \left( \frac{w-w'}{z-w'} \right)^j$. The ratio is less than $1/2$ if $z \in K$.

$$ \frac{|z-w'|}{\rho} < \frac{|w-w'|}{\rho} < \frac{\rho}{2}. $$

Because of this, we can choose $N$ such that $\left| \frac{1}{z-w} - \sum_{j=0}^{\infty} \left( \frac{w-w'}{z-w'} \right)^j \right| < \eta$ for $z \in K$.

Corollary 7.3. For all $\ell$ and $\eta > 0$, there exists $N_\ell$ and $\{a_j, \ell\}$ such that $\left| \frac{1}{z-w'} - \sum_{j=1}^{N_\ell} \frac{a_j}{(z-w')^j} \right| < \eta$ for $z \in K$.

8 Lecture 8

In the homework, for proving the harmonic Runge theorem, assume that $K^c$ is connected, which is stronger than $K$ simply-connected.

Let $K^c$ be connected and hence is just the unbounded component. Let $f$ be holomorphic in a neighborhood of $K$. To show that given $\epsilon > 0$, there exists a polynomial $h(z)$ such that $\|f - h\|_{L^\infty(K)} < \epsilon$.

For all $z \in K$, let $f(z) = \frac{1}{2\pi} \int_{\gamma} \frac{f(w)}{w-z} dw$ for $\gamma \in K^c$. Define $\text{dist}(\gamma, K) = \delta > 0$ and $|w-z| > \delta$ for $z \in K$ and $w \in \gamma$. Then $\left| f(z) - \sum_{j=1}^{N} \frac{f(w_j)(w_{j+1}-w_j)}{(w_j-w_{j+1})2\pi} \right| < \epsilon$

if $z \in K$. 

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2nd observation, if $K \subset D_R(0)$, and $w' \in D_{R+1}(0)$, then \( \frac{1}{w - z} = \frac{1}{w} \sum_{j=0}^{\infty} \left( \frac{z}{w} \right)^j \) converges on $K$. Then \(|z/w'| < R/(R + 1)|$ if $z \in K$.

So there exists $\tilde{N}$ such that \( \left| \frac{1}{w - z} - \sum_{j=0}^{\tilde{N}} \frac{1}{w} \left( \frac{z}{w} \right)^j \right| < \epsilon \) for $z \in K$.

So now chose a $C^1$ path $\ell_j \subset K^c$ such that $\ell_j(0) = w_j$ and $\ell_j(1) = w_j'$ for all $j$.

So there exists a $\rho$ such that $\text{dist}(\ell_j, K) \geq 2\rho$ for all $j$. If $w, w' \in \ell_j$, and $|w - w'| < \rho$, $\epsilon > 0$ given, then there exists \( \{a_j\} \), $N$ such that \( \left| \frac{1}{w - z} - \sum_{j=0}^{N} \frac{a_j}{(w - z)^j} \right| < \epsilon \) for $z \in K$.

Maximum principal implies that if $f$ is holomorphic on $K \cup U$, then $\sup_{z \in U} |f(z)| \leq \sup_{z \in K} |f(z)|$. The sequence $(z - \zeta)p_n(z)$ actually converges to function 1 in $K \cup U$.

Because $(z - \zeta)p_n(z)|_{z = \zeta} = 0$ for all $\zeta \in U$.

8.1 Isolated Singularities

$f$ is holomorphic in $D_r(z_0) \setminus \{z_0\}$. We call this a deleted neighborhood of $z_0$.
There are three types of singularities: removable, poles and essential singularities.

**Theorem 8.1** (Riemann Removable Singularities Theorem). If $f$ is holomorphic in a deleted neighborhood of $z_0$ and $|f(z)|$ is bounded, then $f$ can be extended as a holomorphic function on the neighborhood.

**Proof.** $f(z) = \frac{1}{2\pi i} \int_{bD_r(z_0) \setminus bD_s(z_0)} f(w)dw/(w - z)$. Eh, look this up in Stein-Shakarchi. \( \square \)

Zeros: Suppose that $f$ is holomorphic in $D_r(z_0)$ and vanishes at $z_0$, but not identically. Then there exists an $n$ such that $f^{[j]}(z_0) = 0$ for $j = 0, \ldots, n - 1$ and $f^{[n]}(z_0) \neq 0$. Thus $f(z)/(z - z_0)^n$ is bounded in $D_r(z_0) \setminus z_0$, and so is holomorphic. We say that $f$ has a zero of order $n$ if $f(z) = (z - z_0)^ng(z)$ where $g(z_0) \neq 0$.

Let $f$ be holomorphic in a deleted neighborhood and have infinite limit. Then there exists $\rho > 0$ such that $f(z) \neq 0$ for $z \in D_{\rho}(z_0) \setminus z_0$ or $g = 1/f$ is holomorphic. We have a pole, and poles have form $(z - z_0)^{n}n(z)$ a nonzero holomorphic functions.

Riemann Sphere basics.
Let’s suppose that $f$ is a holomorphic function at $\mathbb{C} \setminus \{z_1, \ldots, z_n\}$ and that $f$ has at worst a pole at $z = \infty$. Then $f(z) = p(z)/q(z)$ for polynomials $p$ and $q$.

9 Lecture 9

**Theorem 9.1** (Residue Theorem). $\frac{1}{2\pi i} \int_{C} \frac{f(w)dw}{(w - z)^{n+1}} = \frac{d^n f}{dz^n}(z)$. 

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Definition of residue
If the pole at $z_0$ is simple, then $\text{Res}_{z_0} f = \lim_{z \to z_0} (z - z_0)f(z)$.
Examples, including the fact that sech($\pi x$) is its own Fourier transform.

**Theorem 9.2** (Casorati-Weierstrass). Suppose that $f$ has an essential singularity at $z = z_0$. Then any punctured neighborhood has image dense in $\mathbb{C}$.

**Theorem 9.3** (Picard’s Great Theorem). If $f$ has an essential singularity at $z_0$, then $f$ assumes every complex value infinitely often except possibly one value.

10 Lecture 10
Some stuff about orientation
Definition of holomorphic manifolds.
Argument Principle.

11 Lecture 11
Define winding number of $\gamma(t)$ a closed curve to be $\int_a^b \gamma'(t)/\gamma(t)dt$.

**Proposition 11.1.** Suppose that $\gamma_t$ is a family of curves not passing through zero. Then $w(\gamma_0) = w(\gamma_1)$.

Proof. $w(\gamma_t) = \frac{1}{2\pi i} \int_a^b \frac{\gamma_t'(s)}{\gamma_t(s)} ds$. $w$ is an integer for all $t \in [0,1]$. Now, as $\gamma_t(s)$ is $C^0$ and $[a,b] \times [0,1]$ is compact, the distance between any curve and 0 is bounded below by a positive number. Thus, $t \mapsto w(\gamma_t)$ is continuous (painful check). A continuous integer value function is constant.

The converse is also true: if $\gamma_0$ and $\gamma_1$ have the same winding number, then they are homotopy equivalent.

Argument principle

**Theorem 11.2** (Rouche’s Theorem). Let $f_t(z)$ be a continuous family of holomorphic functions on $\Omega$, a bounded connected open set with piecewise smooth boundary. $(t,z) \mapsto f_t(z)$ is in $C^0([0,1] \times \Omega)$. Also assume $f_t(z) \neq 0$ on $b\Omega$. Then $\int_{b\Omega} f_t/f'dz$ is constant. Thus, the number of solutions of $f_t(z) = 0$ is independent of $t$.

**Corollary 11.3.** If $f(z)$ and $g(z)$ are holomorphic on $\bar{\Omega}$, and $f(z) \neq 0$ for $z \in b\Omega$, and $|g(z)| < |f(z)|$ for all $z \in b\Omega$, then the number of zeros in $\Omega$ of $f$ is the same as for $f + g$.

The maximum modulus principle is a consequence of the open mapping theorem.
12 Lecture 12

We will review Fourier Series.

Let \( f \in L^2(S^1) \), then \( f(e^{i\theta}) = \sum_{n=\infty}^{\infty} a_ne^{in\theta}/\sqrt{2\pi} \), and define \( <e^{in\theta}/\sqrt{2\pi}, e^{in\theta}/\sqrt{2\pi}> = \delta_{nm} \).

Then \( f_n = (f, e^{in\theta}/\sqrt{2\pi}) \).

To show that the Fourier series converges to \( f \), we will need

**Proposition 12.1** (Bessel’s Inequality). Define \( T_N = \{ \sum_{j=-N}^{N} a_j e^{ij\theta} | a_j \in \mathbb{C} \} \).

If \( f \in L^2(S^1) \), then \( \int_{\mathbb{R}} \|f-t\|_{L^2} \) is attained only at \( S_N(f) = \sum_{j=-N}^{N} f_j e^{ij\theta}/\sqrt{2\pi} \).

\( \|f - t\|_{L^2}^2 = \|f - S_N(t)\|^2 + \|T - S_N\|^2 \).

**Lemma 12.2.** Given \( \epsilon > 0 \), there exists \( N \) and \( t \in T_N \) such that \( \|t - t\|_{L^2} < \epsilon \).

**Lemma 12.3.** If \( f \in L^2(S^1) \) and \( \epsilon > 0 \) is given, then there exists \( g \in C^0(S^1) \) such that \( \|f - g\|_{L^2} < \epsilon \).

**Definition 12.1** (Good Kernel). \( \int_{0}^{2\pi} k_n(x - y) dy = 1, k_n(x + 2\pi) = k_n(x), k_n \in C^{\infty}(S^1), k_n \geq 0, \) and given \( \epsilon > 0, \delta > 0 \), there exist \( N \) such that \( \int_{|x|>1} k_n(x) dx < \epsilon \) if \( n > N \).

Convolution with a good kernel \( k_n * f \rightarrow f \) in \( L^2(S^1) \).

\( \min f \leq k_N * f \leq \max f \).

This reduces to the problem of showing that a continuous function \( g \) can be approximated by trigonometric polynomials in the \( L^2 \)-norm.

Recall the Poisson Kernel \( P(r, \theta, \phi) = \sum_{n=-\infty}^{\infty} r^{n+1} e^{in(\theta - \phi)}/2\pi = \frac{1}{2\pi} \frac{1-r^2}{|1-re^{i(\theta - \phi)}|^2} \).

Define \( g_r(\theta) = \int_{0}^{2\pi} P(r, \theta, \phi) g(\phi) d\phi = \sum r^{n+1} g_n e^{in\theta} \). So there exists \( N \) such that \( \|g_n - g\|_{L^2} \leq 2\epsilon \).

**Proposition 12.4** (Parseval Identity). \( \sum_{n=-\infty}^{\infty} |f_n|^2 = \int_{0}^{2\pi} |f(e^{i\theta})|^2 d\theta \).

If \( f_n \) is a sequence such that \( \sum |f_n|^2 < \infty \) then \( \lim_{n \rightarrow \infty} \sum_{n=-\infty}^{\infty} f_ne^{in\theta}/\sqrt{2\pi} \) exists in \( L^2(S^1) \).

Take the terms with positive \( n \) or \( n = 0 \) to be \( F_+ \) and with negative to be \( F_- \).

Thus, \( f = F_+ + F_- \). \( F_+ \) extends to a holomorphic function on \( D_1 \).

\( F_- \) extends to a holomorphic function on \( D_1 \) \( \cap \) goes to zero as it goes to infinity.

Let \( H^2 \) be the functions of the form \( \sum_{n=0}^{\infty} a_n e^{in\theta} \) with \( |a_n|^2 < \infty \), and let \( H^2 \) analogous.

Then \( L^2(S^1) = H^2 \oplus H^2 \), and \( H^2 \bot H^2 \).

\( H^2(D_1) = \{ f \in L^2(D_1) | f \text{ holomorphic on } D_1 \} \).

It was an exercise to show that the \( L^2 \) functions in the unit disc which are holomorphic define a closed subspace of \( L^2(D_1) \), which we will call \( H^2 \).

Definition of Schwartz Distributions.

\( H^2(D_1) \otimes D_1 \neq H^2(S^1) \).
\[ \mathcal{H}_2(S^1) \text{ is also a closed subspace of } L^2(S^1), \text{ there is an orthogonal projection } \pi_+: L^2(S^1) \rightarrow H^2_+(S^1). \pi_+ \left( \sum_{n=-\infty}^{\infty} f_n e^{in \theta} / \sqrt{2 \pi} \right) = \sum_{n=0}^{\infty} f_n e^{in \theta} / \sqrt{2 \pi}. \]

Is there a "function" \( s(\theta, \phi) \) such that \( \pi_+(f e^{i \theta}) = \int_{0}^{2\pi} f e^{i \theta} s(\theta, \phi) d\phi? \)

If \( f \) is holomorphic on \( D_1 \) and smooth up to the boundary, we know that
\[ f(z) = \frac{1}{2 \pi} \int_{0}^{2\pi} f(e^{i \theta}) e^{-iz \theta} d\theta = \frac{1}{2 \pi} \pi_0^{2\pi} f(e^{i \theta}) \frac{\sin(\frac{\theta - \phi}{2})}{\sin(\frac{\theta - \phi}{2})} d\theta \]

The last term goes to \( \mathcal{C} \int_0^{2\pi} \frac{\cos(\phi - \theta)}{\sin(\phi - \theta)}/2 f(e^{i \theta}) d\theta. \)

So then \( \mathcal{H} f(z) = \Pi_+ f. \)

\[ \lim_{n \to -1} \mathcal{C} f(e^{i \theta}) = \frac{f(e^{i \theta})}{2} + \frac{1}{2 \pi} \int_{0}^{2\pi} \sin \left( \frac{\theta - \phi}{2} \right) f(e^{i \theta}) \frac{\sin(\frac{\theta - \phi}{2})}{\sin(\frac{\theta - \phi}{2})} d\theta \]

This is insane.

Definition 13.1 (Fredholm). \( A \) is Fredholm if \( \text{Range } A \) is closed, \( \ker A \) and \( \text{coker } A \) are finite dimensional.

Then we get \( A^* \) by \( (A^* z, w) = (z, A w). \)

Fredholm Alternative: The condition that \( Ax = y \) has a solution iff \( y \perp \ker A^* \).

A consequence is that \( \dim \ker A - \text{dimker } A^* = n - m. \)

So now we look at \( A : H \rightarrow H, \) an operator on a hilbert space. In infinite dimensional, we might not have the range of \( A \) closed.

13 Lecture 13

\( \mathcal{C} : C^1(bD_1) \rightarrow H(D_1), \) holomorphic functions in the disc.

\[ \mathcal{C} f(z) = \frac{1}{2 \pi} \int_0^{2\pi} f(e^{i \theta}) e^{iz \theta} d\theta. \]

Lemma 13.1. \( \lim_{n \to -1} \mathcal{C} f(z) = \Pi_+ f. \)

\[ \lim_{n \to -1} \mathcal{C} f(e^{i \theta}) = \frac{f(e^{i \theta})}{2} + \frac{1}{2 \pi} \int_0^{2\pi} \sin \left( \frac{\theta - \phi}{2} \right) f(e^{i \theta}) \frac{\sin(\frac{\theta - \phi}{2})}{\sin(\frac{\theta - \phi}{2})} d\theta \]

\[ \mathcal{H} f(z) = \Pi_+ f. \]

For \( f \in C^1(S^1), \) we have \( \| H f \|_{L^2} \leq \| f \|_{L^2}. \) Hence, as \( C^1(S^1) \) is dense in \( L^2(S^1), \) we have \( \mathcal{H} : L^2 \rightarrow L^2. \)

And so, \( \Pi_+ = \mathcal{H} \mathcal{H}^* \mid \mathcal{H}^2. \)

\( \mathcal{H} f \in L^\infty(S^1), \) and so we define \( T_fh = \pi_+(fh) \) is a map \( L^2 \rightarrow H^2, \) and often thought of as \( H^2 \rightarrow H^2. \)

These are called Toeplitz operators, and are Fredholm operators (and compact operators).

Let \( A : C^n \rightarrow C^n. \) When does \( Ax = y \) have a solution?

Define \( (z, w) = \sum z_i \bar{w}_i, \) an inner product.

Then we get \( A^* \) by \( (A^* z, w) = (z, A w). \)

Fredholm Alternative: The condition that \( Ax = y \) has a solution iff \( y \perp \ker A^*. \)

So then, if \( A \) is a fredholm operator, then \( Ax = y \) is solvable if \( y \perp (\ker A^*). \)

We define \( \text{Ind}(A) = \dim \ker A - \dim \text{coker } A. \) This is called the Fredholm index.
Theorem 13.2 (Toeplitz/Noether Index Theorem). The operator $T_f$ is Fredholm provided that $f$ is nonvanishing. Additionally, $\text{Ind}(T_f) = -w(f)$ where $w(f)$ is the winding number.

$$2 = \|T_f h\|^2 = (\Pi_+ f \Pi_+ h, \Pi_+ f \Pi_+ h) \leq f \Pi_+ h\|^2 = \int_0^{2\pi} |f|^2 \Pi_+ h|^2 dx \leq \|f\|_{L^\infty(S^1)}^2 \|\Pi_+ h\|_{L^2} \leq \|f\|_{L^\infty(S^1)}^2 \|h\|_{L^2}^2.$$ 

If $f \in L^\infty(S^1)$, then $\|T_f h\|_{L^2(S^1)} \leq \|f\|_{L^\infty(S^1)} \|h\|_{L^2(S^1)}$.

If $A : X \to Y$ is 1-1 and onto and bounded, then $A^{-1}$ is bounded.

A : $(\ker A)^\perp \to \text{Im } A$ is 1-1 and onto, then the open mapping theorem says that we get an operator $B : \text{Im } A \to (\ker A)^\perp$ which inverts $A$.

Let $P_1$ be the orthogonal projection onto $\text{Im } A$. Set $C = BP_1 : H \to H$ and $AC = ABP_1 = P_1 = I + (P_1 - I)$, while $I - P_1$ is the orthogonal projection onto $(\text{Im } A)^\perp$.

$CA = BP_1 A = BA = P_2$, and so we set $I - P_2 = K_2$ and $I - P_1 = K_1$. Then $AC = I - K_1$, and $CA = I - K_2$, where $K_1$ and $K_2$ are finite rank operators. So Fredholm operators are invertible up to a finite rank error.

Definition 13.2 (Pseudoinverse). Let $A : H \to H$ be bounded. Then $C$ is a pseudoinverse of $A$ if $AC = I - K_1$, $CA = I - K_2$ where $K_1, K_2$ are finite rank.

Theorem 13.3. $A$ is Fredholm iff it has a pseudoinverse.

If $A$ is bounded, we can define $tr A = \sum_{j=1}^\infty (Af_j, f_j)$, where $\{f_j\}$ is an orthonormal basis for $H$.

$tr K_1 = \dim \ker A^*$ and $tr K_2 = \dim \ker A$, and so $\text{Ind}(A) = tr K_2 - tr K_1$ for any choice of pseudoinverse.

An operator is compact if there is a sequence $K_n$ of finite rank operators such that $|||K_n - K||| = \sup_{x \neq 0} \frac{||K_n x - K x||}{||x||} = \Delta_n$ tends to zero as $n \to \infty$.

Proposition 13.4. If $K$ is compact, $\epsilon > 0$, then there is a finite dimensional subspace $S$ such that $||Kx|| \leq \epsilon ||x||$ if $x \perp S$.

Choose a finite rank operator $\tilde{K}$ such that $|||K - \tilde{K}||| < \epsilon$.

If $x \in \ker \tilde{K}$, then $Kx = (K - \tilde{K})x$, and so $||Kx|| \leq ||(K - \tilde{K})x|| \leq ||K - \tilde{K}||||x|| \leq \epsilon ||x||$.

If $||B|| < 1$, then $(I - B)$ is invertible, and so $(I - B)^{-1} = \sum_{j=0}^\infty B^j$. This is called the Neumann series.

14 Lecture 14

$K : H \to H$ is compact if there is a sequence of linear operators of finite rank $K_n$ such that $||K - K_n|| \to 0$ as $n \to \infty$ uniformly convergent.

For all $h \in H$, $\lim_{n \to \infty} K_n h = K h$ is called Strong Convergence.

Let $f \in L^2[0, 1]$, then $\mathcal{F}_N(f)(k) = f_k$ for $|k| \leq N$ and 0 else. For any fixed $f$, we have that $\lim_{N \to \infty} \mathcal{F}_N(f) = < \hat{f}_n >$. But it doesn’t converge uniformly.
An equivalent definition of compact is that $KB_1$ is compact as a subset of $H$.

Let $\mathcal{B}$ be the bounded operators on $H$, $\mathcal{K} \subset \mathcal{B}$ the compact operators. Then $\mathcal{K}$ is a 2-sided ideal in $\mathcal{B}$.

So now let $K_n$ be a finite rank sequence such that $\|K - K_n\| \to 0$ as $n \to \infty$. $\|AK - AK_n\| = \|A(K - K_n)\| \leq \|A\|\|K - K_n\|$.

Let $A : H \to H$ be Fredholm if there is a bounded operator $B : H \to H$ such that $AB = I - K_1$ and $BA = I - K_2$ where $K_1, K_2$ are finite rank.

It suffices to require that $K_1$ compact.

**Proposition 14.1.** If $K : H \to H$ compact and $\epsilon > 0$ given, then there exists a subspace $S \subseteq H$ such that $\dim S < \infty$ and $\|Kx\| \leq \epsilon \|x\|$ for all $x \in S^\perp$.

**Proof.** Choose a sequence $K_n$ of finite rank operators $\|K - K_n\| \to 0$.

There exists $N$ such that $\|K - K_N\| < \epsilon$. Then, if $x \in \ker K_N$, we have $Kx = (K - K_n)x$ so $\|Kx\| \leq \|(K - K_N)x\| \leq \|K - K_n\||x| \leq \epsilon \|x\|$. We can write $K_n = K_{n+1} + K_{n+2}$, where $\|K_{n+1}\| < 1/2$ and $\text{rank}(K_{n+1}) < \infty$. Then $AB = I - K_1 - K_2$ and $BA$ similarly.

So then $(I - K_1)$ is invertible, and so $AB(I - K_1)^{-1} = I - K_1(I - K_1)^{-1}$ (and similarly for $BA$.)

So now we have $AB_1 = I - K_1'$ and $B_2A = 1 - K_2'$.

$A : H \to H$ is Fredholm, then $\text{Ind} A = \dim \ker A - \dim \text{coker} A$.

If $K$ is compact, then $A + K$ is also Fredholm and $\text{Ind}(A + K) = \text{Int}(A)$.

If $A$ is Fredholm, then there exists $\epsilon > 0$ such that if $B : H \to H$ is bounded and $\|B\| < \epsilon$ then $A + B$ is Fredholm, and $\text{Ind}(A + B) = \text{Ind}(A)$.

If $A, B$ are both Fredholm, so is $AB$ and $\text{Ind}(AB) = \text{Ind}(A) + \text{Ind}(B)$.

**Proposition 14.2.** If $K$ is compact, then $I + K$ is Fredholm and $\text{Ind}(I + K) = 0$.

**Proof.** Reduce to $I + \hat{K}$ where $\hat{K}$ is finite rank and then we reduce to a finite dimensional vector space. Suppose that $B$ is pseudoinverse to $A$. Then $BA = I - K_2$ and $AB = I - K_1$. Then $\text{Ind}(AB) = \text{Ind}(A) + \text{Ind}(B) = 0$, and so $\text{Ind}(A) = -\text{Ind}(B)$.

As $A^*A$ is self-adjoint, $\dim \ker A^*A = \dim \ker (A^*A)^* = \dim \text{coker} A^*A$, and so $\text{Ind}(A^*) = 0$, $\text{Ind}A = -\text{Ind}A^*$, if $B$ is a pseudoinverse for $A$, then it is also a pseudoinverse for $A + K$.

Confused

So now look at $H_2^+ \subseteq L^2(S^1)$, where the negative fourier coefficients are zero. $\Pi_+ : L^2(S^1) \to H_2^+$ the orthogonal projection of $f \in C_0(S^1)$, then we define $T_f u = \Pi_+ f \Pi_+ u$ to get $T_f : H_2^+ \to H_2^+$.

**Theorem 14.3 (Szego Index Theorem).** If $f$ is non-vanishing, then $T_f$ is Fredholm and $\text{Ind} T_f = -w(f)$, $f : S^1 \to \mathbb{C} \setminus \{0\}$.

**Proposition 14.4.** $\|\Pi_+ f\|, \|f\Pi_+\| < \|f\|_{L^\infty}$.

More confusion. What the hell is going on?
Lemma 14.5. $[\Pi_+, f] = \Pi_+ f - f \Pi_+$ is always compact.

Proposition 14.6. Let $g \in \mathcal{F}_N$ (operators of rank at most $N$), then $[g, \Pi_+]$ is a finite rank operator.

We still need to compute $\text{Ind} T_f$.

Suppose that $t \mapsto f_t$ defines a continuous map $[0, 1] \times S^1 \to \mathbb{C} \setminus 0$. Then $\text{Ind} T_{f_t}$ is constant.

For fixed $t$, $T_{f_t}$ is Fredholm, and so there exists $\epsilon_t$ such that if $|||T_{f_t} - T_{f_s}||| < \epsilon_t$ then $\text{Ind} T_{f_t} = \text{Ind} T_{f_s}$.

Thus, $\text{Ind} T_{f_t} = \text{Ind} T_{f_s}$ if $|s - t| < r_t$, $t \mapsto T_{f_t}$.

Recall:

Theorem 14.7. If $f$ and $g : S^2 \to \mathbb{C} \setminus 0$ are $C^1$, then $f \simeq g$ iff $w(f) = w(g)$.

Let $n = w(f)$. $f_n(e^{i\theta}) = e^{i n \theta}$ has the same index. And so $\text{Ind} T_{f_n} = \text{Ind} T_f$ must be a function of the winding number.

So we calculate $\text{Ind} \Pi_+ e^{i n \theta} \Pi_+$. If $n = 0$, we get $\text{Ind} \Pi_+ = 0$. If $n > 0$, then ...see problem set.

Thus, $\text{Ind} T_f = -\frac{1}{2 \pi i} \int_0^{2\pi} \frac{f'(e^{i\theta})d\theta}{f(e^{i\theta})}$.

15 Lecture 15

We proved the following:

Theorem 15.1. If $f : S^1 \to \mathbb{C} \setminus 0$, then $T_f = \Pi_+ f \Pi_+ : H^2_+ \to H^2_+$ is Fredholm and $\text{Ind} T_f = -w(f)$.

Theorem 15.2. Let $f : S^1 \to \mathbb{C} - 0$.

1. If $w(f) > 0$, then $T_f$ is 1-1, and so $\text{codim} \text{Im} T_f = w(f)$.

2. If $w(f) < 0$, then $\dim \ker T_f = n$ and $T_f$ is onto.

3. If $w(f) = 0$, then $T_f$ is an isomorphism.

Proof. We will first prove 3. If $w(f) = 0$, then $\log f : S^1 \to \mathbb{C}$ is cont, $\log f = g_+ + g_-$ which are in $H^2_+$ and $(H^2_+)^\perp$ where $g_+$ is the boundary values for a function holo on $D_1$ and $g_-$ is on the complement.

Now take $u \in H^2_+$. Then $f u = u_+ + u_-$, so then $f = e^{g_+}e^{g_-}$, so we have $u = e^{-g_+} \Pi_+ e^{g_-} u_+$, and so $\ker T_f = 0$, and thus, because index is zero, we have $\text{coker} T_f = 0$ as well.

As $T_f^* = T_f$, we only need to prove one of the others. Careful and straightforward computation.
we now extend our notion of "function slightly to line bundles.

Let \( f_+ \in H^2_+ \), \( f_- \in H^2 \) and \( f_+ \phi = f_- \) along the unit circle on the riemann sphere, where \( \phi \) is a map \( S^1 \to \mathbb{C} \). This defined a line bundle \( L_\phi \) and a section \( F : S^2 \to L_\phi \).

Now given \( \phi \in C^1(S^1) \), we want to know if there is a pair with \( f_+ \phi = f_- \).

\[
\lim_{\xi \to \infty} f_-(z) = 0, \text{ then } \Pi_+(f_+ \phi) = \Pi_+(f_-) = 0.
\]

The existence is then equivalent to \( \ker T_0 \neq 0 \), that is, \( w(\phi) > 0 \).

So then there are holomorphic global sections iff \( w(\phi) > 0 \).

If \( w(\phi) = 0 \) then we have \( \mathbb{C} \times \mathbb{P}^1 \), the trivial line bundle.

Up to holomorphic equivalence, holomorphic line bundles are classified by degree (winding number)

We can generalize, taking \( F : S^1 \to \mathbb{C}^n \), then \( \Pi_+ F \) is done componentwise.

A map \( \Phi : S^1 \to gl(n, \mathbb{C}) \) defines a Toeplitz operator by \( T_\Phi = \Pi_+ \Phi \Pi_+ \).

**Theorem 15.3.** \( T_\Phi \) is Fredholm provided that \( \phi : S^1 \to GL(n, \mathbb{C}) \), that is, the determinant is nowhere zero. \( [T_\Phi, \Pi_+] \) is compact.

### 16 Lecture 16

Missed

### 17 Lecture 17

Lidski’s Theorem says that the trace of \( K \) (where \( K \) is integration against \( k(x, y) \) is \( \int_0^{2\pi} k(x, x)dx \).

Suppose that \( K \) is diagonalizable and there exists an orthonormal basis such that the span is \( L^2(S^1) \) and \( Ke_j = \lambda_j e_j \). Then \( \text{tr } K = \sum_{j=1}^{\infty} \lambda_j = \int_0^{2\pi} k(x, x)dx \).

So \( K f = \int_0^{2\pi} \phi(x - y) f(y) dy \) where \( \phi \) is a \( 2\pi \)-periodic \( C^1 \)-function. Computation that gives Fourier Inversion Formula.

Chosoe a function \( f \in C^1(\mathbb{R}) \) such that \( |f(x)| \leq A/(1 + x^2) \) and \( |f'(x)| \leq A/(1 + x^2) \). Then set \( \phi(x) = \sum_{n=-\infty}^{\infty} f(x + n) \). Standard stuff that is in Stein-Shakarchi.

**Theorem 17.1** (Paley-Wiener). A function \( f \) with moderate decrease along \( \mathbb{R} \), has an extension to \( \mathbb{C} \) as an entire function satisfying \( |f(z)| \leq Ce^{2\pi M|z|} \) if and only if \( \hat{f}(\xi) \) is supported in \( [-M, M] \).

We showed last time that if the support of the Fourier transform is compact, then there is an analytic extension as desired. Now we suppose \( f(z) \) entire and \( |f(x + iy)| \leq Ce^{2\pi M|y|} \) and \( |f(x + iy)| \leq A/(1 + x^2) \).

We can compute \( \hat{f}(\xi) = \int_0^\infty f(x + iy)e^{-2\pi i(x + iy)\xi}dx = \int_0^\infty f(x + iy)e^{-2\pi i\xi x}e^{2\pi i\xi y}dx \)

case \( \xi < -M \).  

Thus \( |\hat{f}(\xi)| \leq \int_{-\infty}^\infty \frac{Ce^{2\pi M|y|}e^{2\pi \xi y}}{1 + \xi^2}dy \leq Ce^{2\pi (M - \xi)|y|} \) goes to zero as \( y \to \pm \infty \) and so \( |\hat{f}(\xi)| = 0 \) for \( |\xi| > M \).

We start removing hypotheses.
Given $|f(z)| \leq Ae^{2\pi M|z|}$ with $|f(x)| \leq B/(1 + x^2)$, we need to have $|f(x + iy)| \leq A e^{2\pi M|y|}$.

We use that Phragmén-Lindelöf Principle: $f$ is holomorphic in $S$ and continuous up to $\partial S$ with $|f(z)| \leq 1$ for $z \in \partial S$. If for some constants $c, C$, we have $|f(z)| \leq C e^{c|z|}$, then we have $|f(z)| \leq 1$ for all $z \in S$.

Define $F(z) = f(z) e^{-\epsilon z^2/2}$, $z = re^{i\theta}$. So through some calculations, we have $|F(z)| \leq e\epsilon |z|^{1/2}$. So the result follows.

We in fact don’t need $Ce^{c|z|}$, we can use $Ce^{c|z|^\gamma}$ for any $\gamma < 2$.

So now we want to show that $f(z) \leq A/(1 + x^2)$ and $|f(z)| \leq C e^{2\pi M|z|}$ implies that $|f(z)| \leq C e^{2\pi M|y|}$.

Look at $F(z) = f(z) e^{2\pi i M y}$. Then $F(x) = f(r) e^{2\pi i M y}$ and so $|F(x)| \leq A/(1 + x^2)$ and $F(i y) = f(i y) e^{-2\pi M y}$. Then $|F(i y)| \leq C e^{2\pi M y}$.

So then Phragmén-Lindelöf says that $|F(z)| \leq C$ in the quadrant, and so $|F(x + iy)| \leq C$ gets $y \geq 0$. By looking at $f(z) e^{-2\pi M z}$ we get the result on the bottom half plane.

So what did we use? $f$ is entire and bounded on $\mathbb{R}$, and $|f(z)| \leq Ce^{2\pi M|z|}$ which gives us that $|f(x + iy)| \leq \tilde{t}de^{2\pi M|y|}$.

**Theorem 17.2** (Hadamard Three Lines Theorem). Let $f$ be defined in a strip $S$ and continuous up to $\partial S$ with $|f(z)| \leq 1$ on each boundary component with $|f(z)| \leq Ce^{c|z|^\gamma}$ for $\gamma < 2$, then $|f(z)| \leq 1$ in $\tilde{S}$.

**Proof.** Look at $f(z) e^{-\epsilon z^2}$. As $z^2 = x^2 - y^2 + 2i xy$, we have $|e^{-\epsilon z^2}| = e^{-\epsilon (x^2 - y^2)} \leq Ce^{-\epsilon x^2}$.

$|f(z) e^{-\epsilon z^2}| \leq C e^{c|z|^\gamma - \epsilon x^2}$, and thus we have shown that the limit as $|z| \to \infty$ of this is zero for all $\epsilon > 0$. The maximum principle says that there exists $w$ with $\Im(w) = 0$ on 1 and $|f(w)|e^\epsilon \geq |f(w) e^{-\epsilon w^2}| \geq |f(z) e^{-\epsilon z^2}|$ for all $z \in S$.

**18 Lecture 18**

Hadamard needed his three lines theorem to prove the Prime Number Theorem. He used it to get some estimates.

**Theorem 18.1** (Hadamard). Let $f$ be a bounded analytic function in $0 \leq \text{Res} \leq 1$ and let $N(a) = \sup_{y \in \mathbb{R}} |f(a + iy)|$, then $N(a) \leq N(0)^{1-a} N(1)^a$.

**Corollary 18.2.** $\log N(a)$ is a convex function.

If $(X, dm)$ and $(Y, dn)$ are measure spaces, look at $L^p(X, dm) = \{f \text{ measurable with } (\int_X |f|^p dm)^{1/p} < \infty\}$. Let $F$ be a linear operator and carry measurable functions on $X$ to measurable functions on $Y$. Then $F(f) = \int_0^{2\pi} f(\theta) e^{-i n \theta} d\theta = \hat{f}(n)$ takes measurable functions on $S^1$ to measurable functions on $\mathbb{Z}$.

Hadamard implies that for $1 < p < 2$, we have $L^p(S^1) \to \ell_\infty$ and $L^2(S^1) \to \ell_2$.

To show this, we define $F : L^p(X, dm) \to L^q(Y, dn)$ is bounded if $\sup_{x \neq 0} \|F(x)\|_{L^q(dm)} / \|x\|_{L^p(dm)} = M(p, q) < \infty$. 

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So then \( \|F(x)\|_{L^q(dm)} \leq M(p, q)\|x\|_{L^p(dm)} \).

**Theorem 18.3** (F. Riesz-Markov). \((L^p(X, dm))^\vee \cong L^{p'}(X, dm)\) for \(1/p + 1/p' = 1\), and also \( \|f\|_{L^p(dm)} \leq (\int |f|^p dm)^{1/p} \left( \int |g|^{p'} dm \right)^{1/p'} \).

A consequence is that \( f \mapsto (f, g) = \int f g dm \) gives \( \|f\|_{L^p(X, dm)} = \sup_{g \neq 0} (f, g)/\|g\|_{L^{p'}(X, dm)} \).

**Theorem 18.4** (N. Riesz Interpolation Theorem). Suppose that \( M : L^p(X, dm) \to L^q(Y, dn) \) and \( M : L^{p'}(X, dm) \to L^{q'}(Y, dn) \) are bounded. Then \( M : L^{p(a)}(X, dm) \to L^{q(a)}(Y, dn) \) for \((1/p(a), 1/q(a)) = (1 - a)(1/p_0, 1/q_0) + a(1/p_1, 1/q_1)\) satisfies \( M(p(a), q(a)) \leq M^{1-a}(p_0, q_0)M(p_1, q_1)^a \).

This would apply, for instance, to the Fourier transform from \( L^2(S^1) \) to \( \ell_2 \) and \( L^1(S^1) \) to \( \ell_\infty \).

**Proof.** For \( a = 0 \) or \( a = 1 \), the estimate is trivial, so choose \( 0 < a < 1 \).

Choose a function \( f \in L^p(X, dm) \) and \( h \in L^{q'}(Y, dn) \) such that \( Mf \) is defined and belongs to \( L^q \).

\[ f = |f|e^{ith} \quad \text{and} \quad h = |h|e^{i\nu} \]

Then \( f(\zeta) = |f|^{p(a)/p}(\zeta)e^{i\theta(\zeta)} = h^{q(a)/q}(\zeta)e^{i\nu}, \]

where \( 1/p(a) = (1 - a)/p_0 + a/p_1 \).

Define \( \phi(\zeta) = (Mf(\zeta), h(\zeta)) = \int_Y Mf(\zeta)h(\zeta)dn. \)

Then \( \phi(\zeta) \) is analytic in the strip. Suppose that \( \|f\|_{p(\zeta)} = 1 \) and \( \|h\|_{q(\zeta)} = 1 \). Then if \( N(a) = \sup_{\eta \in \mathbb{K}} |\phi(a+i\eta)| \) then \( N(0) \leq M(p_0, q_0) \) and \( M(1) \leq M(p_1, q_1) \).

After a nasty computation, we see that \( \|Mf(i\eta)\|_{L^q} \leq M(p_0, q_0)\|f\|_{L^p} \leq M(p_0, q_0). \)

We conclude that \( |\phi(i\eta)| = |(Mf(i\eta), h(i\eta))| \leq \|Mf(i\eta)\|_{L^q} \|h(i\eta)\|_{q_0'} \leq M(p_0, q_0). \)

\( |\phi(1+i\eta)| \leq M(p_1, q_1). \) Now, \( \phi(\zeta) \) satisfies the hypotheses of Hadamard’s theorem, and so \( |\phi(a)| \leq M(p_0, q_0)^{1-a}M(p_1, q_1)^a. \)

We recall that \( \phi(\alpha) = (Mf, h) \). We note that \( \|Mf\|_{L^q} = \sup \|(Mf, h)\| \) over \( h \) with \( \|h\| = 1 \).

A bit more computation proves the theorem.

Now that we can interpolate between \( L^1(S^1) \) and \( L^2(S^1) \), we will return to Paley-Wiener type theorems.

If we assume that \( |f(x)| \leq \frac{C}{(1+|x|)^N} \), then this estimate allows us to differentiate the integral \( \hat{f} \) up to \( N - (1 + \epsilon) \) times. (we can only just barely not do it \( N - 1 \) times) gibberish...

**Theorem 18.5.** Let \( f \in L^2(\mathbb{R}) \), then teh following two conditions are equivalent:

1. There exists a holomorphic \( F(x + iy) \) for \( y > 0 \) with \( \sup_{y \geq 0} \int_{-\infty}^{\infty} |F(x + iy)|^2 dx < C \) and \( \lim_{y \to -0} \int_{-\infty}^{\infty} |F(x + iy) - F(x)|^2 dx = 0 \)

2. \( \hat{f}(\xi) = 0 \) for \( \xi < 0 \).

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19  Lecture 19

We will prove the theorem from last time. 2 implies 1 is easy. See Stein-Shakarchi

**Corollary 19.1.** $L^2(\mathbb{R}) = L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R})$ where $L^2_+(\mathbb{R})$ are the functions which satisfy the first part of the previous theorem.

To see that $L^2_+ \cap L^2_- = 0$, we note that the Fourier transform takes the first to $L^2((0, \infty))$ and the second to $L^2((-\infty, 0))$.

For $x \in [0, \infty)$, we define the Wiener-Hopf equation to be $\int_0^\infty k(x-y) f(y) dy + \lambda f(x) = g(x)$.

**Theorem 19.2.** Suppose that $u$ is Hölder continuous of order $\alpha > 0$ and that $u \in L^1 \cap L^p$ for $1 \leq p < \infty$. Define $U(z) = \frac{1}{2\pi i} \int_{-\infty}^\infty u(\xi) d\xi/(\xi - z)$. Then $U$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.

To use this to solve the Wiener-Hopf equation, replace $g$ by $g + h$, and then extend $f, g$ by zero to $(-\infty, 0)$, and set $h(x)$ to be zero for $x > 0$ and $\int_0^x k(x-y) f(y) dy$ if $x < 0$.

Now we take Fourier transforms and obtain $\hat{k}(\xi) \hat{f}(\xi) + \lambda \hat{f}(\xi) = \hat{g}(\xi) + \hat{h}(\xi)$.

Now $\hat{f}$ and $\hat{g}$ have analytic extensions to $H_-$ and $\hat{h}$ has an analytic extension to $H_+$.

Gibberish

20  Lecture 20

Very confused about integral equations.

Mittag-Leffler Problem: I specify points $\{a_j\} \subset \mathbb{C}$ and polynomials $p_j(z)$ without constant term. Does there exist a function $f$ holomorphic in $\mathbb{C} \setminus \{a_j\}$ with $f - p_j(1/(z - a_j))$ is holomorphic near to $a_j$ with $\lim_{j \to \infty} |a_j| = \infty$?

Recall that we’ve shown that if $\phi \in C^\infty(\mathbb{C})$ then there is a function $u$ such that $\partial \phi u = \phi$.

Choose $0 < r_j$ such that $D_{2r_j}(a_j) \cap D_{2r_j}(a_k) = \emptyset$ for all $j \neq k$.

Define $\phi$ to be the partial with respect to $z$ of this function. Use teh existence theorem to solve $\partial \phi u = \phi$.

So there exists a $C^\infty$ function $u$ which solves this equation.

Then we define $f(z)$ to be the original function minus $u$.

Clearly, $\partial f = 0$ in $\mathbb{C} \setminus \{a_j\}$, and near to $z = a_j$ we have $\hat{f} u = 0$, so near to $a_j$, we have $f(z) = p(1/(z - a_j)) - u(z)$, so we have $\sum_{j=1}^N p_j(1/(z - a_j))$.

Let $p(z)$ be degree $d$ and $N(r) = \sup_{0 \leq \theta \leq 2\pi} |p(re^{i\theta})|$ and $\lim_{r \to \infty} \log N(r)/\log r = d$, where $d$ is the number of solutions to $p(z) = 0$.

**Theorem 20.1** (Jensen’s Formula). Let $\Omega$ be an open set containig $D_R$ and let $f$ be holomorphic in $\Omega$ such that $f(0) \neq 0$, $f$ does not vanish on $bD_R$ and $\{z_1, \ldots, z_N\}$ are the zeroes of $f$ in $D_R$, listed with multiplicity. Then
\[ \log |f(0)| = \sum_{k=1}^{N} \log \left( \frac{|z_k|}{R} \right) + \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(Re^{i\theta})| d\theta \]

**Proof.** Suppose that \( g \) does not vanish on \( \bar{D}_R \). Then this reduces to the mean value theorem.

Let \( f(z) = \prod_{j=1}^{N} (z - z_j) g(z) \). Then \( \int_{0}^{2\pi} \log |f(Re^{i\theta})| d\theta / 2\pi = \sum_{j=1}^{N} \int \log |Re^{i\theta} - z_j| d\theta / 2\pi + \log |g(0)| \). See book.

We define \( n_f(r) = n(r) \) to be the number of zeroes of \( f \) less than \( r \).

Then \( \int_{0}^{R} n(R)/rdr \) will be a step function counting the zeroes.

**Definition 20.1 (Finite Order).** An entire function with finite order of growth \( \rho \) satisfies the estimate \( |f(z)| \leq Ae^{B|z|^\rho} \).

Gibberish which is in the book.

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**21 Lecture 21**

If \( f \) is entire and \( |f(z)| \leq Ae^{B|z|^\rho} \), and \( n(r) \) is the number of zeroes of modulus less than \( r \) with multiplicity, then \( n(r) \leq Cr^\rho \) for large \( r \).

Given \( (z_j, n_j) \), with \( z_j \in \mathbb{C} \) and \( n_j \in \mathbb{N} \), with \( z_j \to \infty \) as \( j \to \infty \), does there exist a holomorphic function \( f \) defined in \( \mathbb{C} \) such that \( f \) vanishes only at the \( z_j \) and the order of the zero at \( z_j \) is \( n_j \)?

Read the book. Seriously, everything’s there.

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**22 Lecture 22**

Missed some asymptotic stuff

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**23 Lecture 23**

Riemann Mapping Theorem

**Theorem 23.1 (Riemann Mapping Theorem).** If \( D \subseteq \mathbb{C} \) is a simply connected domain, then there exists a holomorphic map \( f : \mathbb{D} \to D \) which is 1-1 and onto.