Interested in decidability & complexity questions in enumerative combinatorics.

Interested mainly in asymptotic results

Example (Wilf 1982) Conjecture: \# unlabelled graphs on \( m \) vertices cannot be computed in \( \text{poly}(n) \) time.

But Erdős–Rényi implies \( \sim 2^n \).

Basic object: \( \text{GF} \ F(z) = \sum a_n z^n \).

In order to hope for decidability, need structure on our objects.

For constants

For GFs

\( \text{Polys} \): Everything trivial

\( \text{D-algebraic} \): Lots really hard

Example: Dumas & Lipshitz '89

Given analytic \( f(z) \) sol. of ADE, undecidable in general to say if radius of convergence is \( <1 \) or \( >1 \).

There are "effective rings" (zero testing, etc.)

\( \mathbb{Z}, \mathbb{Q} \) elements stand implicitly (min poly + isotopy regions, etc.)

\( \text{Polys} \):
- Rational functions
- Algebraic functions
- D-finite (linear ODEs w/ poly coeffs)
- D-algebraic

Interesting, but need different tools (complexity, etc.).
This ranking allows us to give a measure of "complexity" to sequences. Compare to Chomsky hierarchy of formal languages.

Basics of Analytic Combinatorics

Let \( f(z) \) be analytic at 0, \( f(z) = \sum_{n=0}^{\infty} a_n z^n \). Cauchy's root test (c. 1821) implies exponential growth.

\[
P := \limsup_{n \to \infty} \left| \log a_n \right| / \log n
\]

is the smallest modulus of singularity of \( f(z) \).

Ex: \[
\frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n \quad (\text{sing at } \frac{1}{2})
\]

\[
\frac{1 - \sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} \frac{1}{n+1} 2^n \quad (a_n \sim \frac{4^n}{n^{3/2}})
\]

Flajolet & Sedgewick Principle #1: Location of singularities determines exponential growth.

Flajolet & Sedgewick Principle #2: The type of singularity (pole, branch cut, etc.) determines subexponential growth.

Pringsheim's Theorem: If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then one of the dominant singularities is true and real (sufficient to check any \( R_{\infty} \) until hit one).
Most asymptotic results follow from Cauchy residue theorem:

If \( f(z) = \frac{\xi}{n!} a_n z^n \), then

\[
\alpha_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} \, dz,
\]

\( C \) the oriented circle of sufficient small radius.

Also useful:

\[
\left| \int_C g(z) \, dz \right| \leq \text{length}(C) \cdot \max_{z \in C} |f(z)|
\]

**Example:** André (1880s) showed \( \tan(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \cdot x^n \), where \( a_n \)

is the number of alternating permutations for \( n \) even.

Poles of \( \tan(z) \) at \( \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \)

Thus, if \( t_n = [z^n] \tan(z) \),

\[
t_n = \frac{1}{2\pi i} \oint_C \frac{\tan(z)}{z^{n+1}} \, dz.
\]

Also, \( \left| \sum_{k=2}^{\infty} \frac{\tan(z)}{z^{n+1}} \right| \leq 4\pi \cdot z^{-n-1} \cdot \max_{k \geq 2} |\tan(z)| = O(z^{-n}) \) can be anything in \((\frac{\pi}{6}, \frac{\pi}{3})\).

Thus, \( t_n = \frac{1}{2\pi i} \left( -\oint_{C_1} \frac{\tan(z)}{z^{n+1}} \, dz \right) + O(z^{-n}) \)

\[
= - \left[ \text{Res} \left( \frac{\tan(z)}{z^{n+1}} \right) + \text{Res} \left( \frac{\tan(z)}{z^{n+1}} \right) \right] + O(z^{-n})
\]

\[
= \begin{cases} \frac{4}{\pi} \cdot \left( \frac{2}{\pi} \right)^n, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}
\]
This can be greatly extended.

**Thm (F&S IV.10)** Suppose $F(z)$ analytic on $\{z : |z| = R\}$ and contains only poles at $\sigma_1, \ldots, \sigma_m$ in $\{z : |z| < R\}$. Then there exist polynomials $P_1(n), \ldots, P_m(n)$ such that

$$f_n = \sum_{j=1}^{m} P_j(n) \cdot \sigma_j^{-n} + O(R^{-n})$$

(degree 1 less than the order of pole $\sigma_j$)

Similar analysis can be done for other classes of GFS.

The decidability of asymptotics largely comes down to computing the dominant singularities (potentially up to constant).

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**Rational Functions**

A sequence is \(C\)-finite of order \(r\) over the field \(K\) if \(f \in \mathbb{F}_{0, \ldots, r}\) and uniquely specified by first \(r\) terms. (Ex: Fibonacci)

De Moivre (1718, Doctrine of Chance) introduced GFS to study "recurrent" sequences. 

**Prop (De Moivre, 1780)**: \(F(z) = \sum_{k=0}^{\infty} f_k z^k\) is rational if and only if \(f_n = \text{polynomial of degree} < r\).
The zones of $(a_n)$ are $Z_i = \ldots$

There are still interesting decidability questions.

$P_i$ can be computed in $P^P$ for $\Phi_3$.

Let $\alpha$, $\beta$, $\gamma$, $\delta$ be roots of $H_1$.

$F(x) = G(x)$

$F(2) = G(2)$

(See also: AECF, Boston 1987)
"It's like saying we don't know how to decide the halting problem even for linear automata!" — Terry Tao

(Dick Lipton: mathematical embarrassment / Tao: faintly outrageous)

Mignotte, Shorey, Tijdeman (1984) — decidable for orders 4

Vassilvitskii (1985) — decidability for orders 5

Rachhod, Schulze (2002) — decidability for orders 6

Gollin, Grillet (2006) — decidability for orders 7

Use Baker's theorem on linear combs of logs of alg #s

(Fields medal 1970)

Can reduce to positivity as $(p_n^2)$ is C-finite if $(f_n)$ is.

Open Problem: Is it decidable if $(f_n)$ is ultimately positive?

Burke & Webb (1981) — orders 2 decidable

Lachakosol & Tangsupphathum (2009) — orders 3 decidable

Ouknine & Warrilow (2014) — orders 5 decidable

— any orders decidable if domain square-free