# Combinatorics 1 – Problem Sheet 2

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**Exercise 1**

Consider the distance on formal power series discussed in class, $d(F,G) = 2^{-\text{val}(F-G)}$. Prove that the set of convergent power series is dense in the set of formal power series with coefficients in $\mathbb{R}$. In other words, show that for any formal power series $F \in \mathbb{R}[[z]]$ there is a sequence of convergent series (in the usual calculus sense) whose limit is $F$.

**Exercise 2**

Prove that this distance satisfies the strong triangle inequality,

$$d(F,G) \leq \max\left[d(F,H), d(H,G)\right]$$

for any formal power series $F, G, H$. This shows that formal power series form a non-archimedean metric space.

**Exercise 3**

Let $A$ be a commutative ring where all natural numbers have a multiplicative inverse. The formal power series logarithm is defined by

$$\log(1+x) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n \in A[[x]].$$

If $F \in A[[x]]$ is a formal power series with no constant term, show how to compute the first $N$ coefficients of $G(x) = \log(1+F)$ using only $O(M(N))$ multiplications in $A$, where $M(N)$ is the cost of multiplying two polynomials in $A[x]$ of degree $N$. *Hint:* In class we showed that the multiplicative inverse of a power series can be computed using only $O(M(N))$ multiplications in $A$. Take derivatives (you may use that the chain rule still holds for formal derivatives).

**Exercise 4**

In class we saw that the number of plane trees on $n+1$ vertices equals the number of Dyck paths of length $n$ by proving that they have the same generating function. Find a bijection between these classes (find a 'natural' way to pair up each plane tree of length $n+1$ with a Dyck path of length $n$).

**Exercise 5**

Fix integers $m, k \geq 1$. Prove that the generating function $F(z)$ counting words (finite sequences) on an alphabet $\mathcal{A} = \{a_1, \ldots, a_m\}$ with $m$ letters that do not contain $k$ consecutive occurrences of $a_1$ is

$$F(z) = \frac{1 - z^k}{1 - mz + (m-1)z^{k+1}}.$$

Try an induction, or using the constructions discussed in class.