(Practice) In Class MidTerm Exam for Math 113
10 - 11:30 am, March 20, 2008

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- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- Some problems will be asking you to prove results covered in class. For those that do not, you may quote without proof the theorems/propositions/lemmas given during lecture as long as you state each result clearly. Points may be deducted for incorrect or missing hypotheses.
- **NO** calculators are permitted.
- **This is a practice exam.**
1. (a) Give the definition of a conformal map.
(b) Give the condition(s) for a holomorphic map to be conformal.
(c) Consider the wedge $\Omega = \{ z \in \mathbb{C} \mid 0 < \arg(z - 3i) < \frac{\pi}{4} \}$. Give a conformal mapping of $\Omega$ into the unit disk $B_1(0) = \{ z \in \mathbb{C} \mid |z| < 1 \}$ and justify your answer.
2. Suppose that \( a, b \in \mathbb{C} \) are two vertices of a square. Find the two other vertices in all possible cases.
3. Let $A$ be the complement in $\mathbb{C}$ of the non-positive real axis, i.e. $A = \mathbb{C} - \{x \in \mathbb{R} \mid x \leq 0\}$.

(a) Show that there exists a unique function $f$, which is holomorphic on $A$ and satisfies $f(x) = x^x$ for all real $x > 0$.

(b) Find $f(i)$ and $f'(i)$. 
4. (a) Prove or disprove: There exists a holomorphic mapping from \( \mathbb{C} \) to the upper half plane
\[
\{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}.
\]

(b) Prove or disprove: The image of \( \mathbb{C} \) under a nonconstant entire mapping is dense in \( \mathbb{C} \).
5. Consider the stereographic projection of the sphere of radius 1 centered at the origin in $\mathbb{R}^3$

\[ \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \} \]

from the south pole $(0,0,-1)$ onto the $xy$-plane. Derive the expression for the image of $(x_1, x_2, x_3)$ under this map and write it out as a complex number.
6. Let $f$ be entire and $u(x, y) = \text{Re}(f(x + iy))$. Suppose that

$$u(x, y) = u(-y, x).$$

Prove that for all $z \in \mathbb{C}$, $f(z) = f(iz)$. 
7. Show that if $F$ is analytic on $A$, then so is $f$ where

$$f(z) = \frac{F(z) - F(z_0)}{z - z_0},$$

if $z \neq z_0$ and $f(z_0) = F'(z_0)$ where $z_0$ is some point of $A$. 
Extra space for work: