Proof. By our formula on the indefinite integral of \( e^x \sin x \), \( \langle f_1, f_2 \rangle = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^x \sin x dx = e^{\frac{3\pi}{4}}(\frac{\sin x - \cos x}{2}) \bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = e^{\frac{3\pi}{4}}(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) - e^{\frac{\pi}{4}}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) = 0. \) So \( e^x \) and \( \sin x \) are orthogonal on \([-\frac{\pi}{4}, \frac{3\pi}{4}]\).

\[ \square \]

12.1.18

Proof. \( \langle x + c_1 x^2 + c_2 x^3, x \rangle = \langle x, x \rangle + c_1 \langle x^2, x \rangle + c_2 \langle x^3, x \rangle \), and \( \langle x, x \rangle = \int_{-2}^{2} x^2 dx = \frac{1}{3}(8 - (-8)) = \frac{16}{3} \), \( \langle x^2, x \rangle = 0 \) since \( x^2 \cdot x = x^3 \) is an odd function, its integral on \([-2, 2]\) must be zero, and
\[
\langle x^3, x \rangle = \int_{-2}^{2} x^4 dx = \frac{1}{5}(32 - (-32)) = \frac{64}{5}. \]

So \( \langle x + c_1 x^2 + c_2 x^3, x \rangle = \frac{16}{5} + \frac{64}{5} c_2 \). To make \( x + c_1 x^2 + c_2 x^3 \) orthogonal to \( x \), we want this to be zero, so \( c_2 = -\frac{16}{32} = -\frac{1}{2} \).

\[
\langle x + c_1 x^2 + c_2 x^3, x^2 \rangle = \langle x, x^2 \rangle + c_1 \langle x^2, x^2 \rangle + c_2 \langle x^3, x^2 \rangle \geq \langle x, x^2 \rangle = 0 \] since their product functions are odd functions, and \( \langle x^2, x^2 \rangle = \int_{-2}^{2} x^4 dx = \frac{64}{5}. \)

So \( \langle x + c_1 x^2 + c_2 x^3, x^2 \rangle = \frac{64}{5} c_1 \). To make \( x + c_1 x^2 + c_2 x^3 \) orthogonal to \( x^2 \), we want this to be zero, so \( c_1 = 0 \).

\[ \square \]

12.1.21

Proof. (b) \( \sin x \)'s fundamental period is \( 2\pi \), so \( \int_{\frac{\pi}{2}}^{\frac{\pi}{2} + 2\pi} x = \sin(\frac{\pi}{2} x + 2\pi) = \sin \frac{\pi}{2}(x + \frac{\pi}{2} \pi) = \sin \frac{\pi}{2}(x + \frac{\pi}{2}). \) So the fundamental period is \( \frac{\pi}{2} \). In general, \( \sin \omega x \cos \omega x \) is a fundamental period \( \omega x \).

(e) \( \sin 3x \) has fundamental period \( \frac{3\pi}{4} \), \( \cos 2x \) has fundamental period \( \pi \), so their sum should have fundamental period as the least common multiple of \( \frac{3\pi}{4} \) and \( \pi \). If we assume this number is \( T \), then there should be two integers \( m, n \) such that \( T = m \cdot \frac{3\pi}{4} = n \cdot \pi \). This yields \( \frac{m}{n} = \frac{3}{2} \). The smallest positive integers \( m, n \) that satisfy this is \( m = 3, n = 2 \), therefore \( T = 2\pi \) is the fundamental period of \( \sin 3x + \cos 2x \).

\[ \square \]

12.2.13

Proof.
\[
a_0 = \frac{1}{5} \int_{-5}^{0} dx + \int_{0}^{5} (1 + x) dx = \frac{1}{5}[5 + (x + \frac{x^2}{2})]_0^5 = \frac{1}{5}[5 + 5 + \frac{25}{2}] = 2 + \frac{5}{2} = \frac{9}{2}
\]
\[
a_n = \frac{1}{5} \int_{-5}^{0} \cos \frac{n\pi}{5} x dx + \int_{0}^{5} (1 + x) \cos \frac{n\pi}{5} x dx = \frac{1}{5} \int_{-5}^{0} \cos \frac{n\pi}{5} x dx + \int_{0}^{5} \cos \frac{n\pi}{5} x dx + \int_{0}^{5} x \cos \frac{n\pi}{5} x dx
\]
\[
= \frac{1}{5} \int_{-5}^{5} \cos \frac{n\pi}{5} x dx + \int_{0}^{5} x \cos \frac{n\pi}{5} x dx
\]
The first term is zero by period argument, and we calculate the second term by integration by parts:

\[
\frac{1}{5} \int_0^5 x \cos \frac{n \pi}{5} x \, dx = \frac{1}{5} \int_0^5 x \frac{n \pi}{5} \sin \frac{n \pi}{5} x \, dx = \frac{1}{n \pi} \int_0^5 x \sin \frac{n \pi}{5} x \, dx
\]

\[
= \frac{1}{n \pi} \left[ (x \sin \frac{n \pi}{5} x) \bigg|_0^5 - \int_0^5 \sin \frac{n \pi}{5} x \, dx \right] = -\frac{1}{n \pi} \int_0^5 \sin \frac{n \pi}{5} x \, dx = \frac{1}{n \pi} \cos \frac{n \pi}{5} x \bigg|_0^5 = \frac{5}{n^2 \pi^2} \cos n \pi - 1 = \frac{5((-1)^n - 1)}{n^2 \pi^2}
\]

Similarly,

\[
b_n = \frac{1}{5} \left[ \int_{-5}^0 \sin \frac{n \pi}{5} x \, dx + \int_0^5 (1 + x) \sin \frac{n \pi}{5} x \, dx \right] = \frac{1}{5} \left[ \int_{-5}^5 \sin \frac{n \pi}{5} x \, dx + \int_0^5 x \sin \frac{n \pi}{5} x \, dx \right]
\]

The first term is zero by period argument, and we calculate the second term by integration by parts:

\[
\frac{1}{5} \int_0^5 x \sin \frac{n \pi}{5} x \, dx = \frac{1}{5} \int_0^5 x \frac{n \pi}{5} \cos \frac{n \pi}{5} x \, dx = -\frac{1}{n \pi} \int_0^5 x \cos \frac{n \pi}{5} x \, dx
\]

\[
= -\frac{1}{n \pi} \left[ (x \cos \frac{n \pi}{5} x) \bigg|_0^5 - \int_0^5 \cos \frac{n \pi}{5} x \, dx \right] = -\frac{1}{n \pi} \left[ 5 \cos n \pi - \frac{5}{n \pi} \sin \frac{n \pi}{5} x \bigg|_0^5 \right] = -\frac{5((-1)^n)}{n \pi}
\]

Therefore the Fourier series for \( f(x) \) is

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n \pi}{5} x + b_n \sin \frac{n \pi}{5} x = \frac{9}{4} + \sum_{n=1}^{\infty} \frac{5((-1)^n - 1)}{n^2 \pi^2} \cos \frac{n \pi}{5} x - \frac{5((-1)^n)}{n \pi} \sin \frac{n \pi}{5} x
\]

\( \square \)

**Fall 11, 9**

**Proof:**

\[
a_0 = \frac{1}{\pi} \int_0^\pi dx = 1
\]

\[
a_n = \frac{1}{\pi} \int_0^\pi \cos nx \, dx = \frac{1}{n \pi} \sin nx \bigg|_0^\pi = 0
\]

so all the \( a_k \)'s are zero, therefore \( a_0 + \sum_{k=1}^{\infty} a_k^2 b_k = a_0 = 1 \).

\( \square \)