Category Theory in Geometry

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Categories

Category: a collection of objects and morphisms between objects

- Every object \( c \) has an identity morphism \( I_c \)
- For morphisms \( f : c \to d \) and \( g : d \to e \), there is a composite morphism \( gf : c \to e \)

Examples:
- Sets & functions
- Groups & group homomorphisms
- Topological spaces & continuous functions
An **isomorphism** is a morphism $f : c \to d$ with $g : d \to c$ so that $fg = I_d$ and $gf = I_c$.

**Examples**
- **Set**: bijections
- **Group**: group isomorphisms
- **Top**: homeomorphisms
Functors

**Functor:** a map $F : C \rightarrow D$ between categories taking objects to objects and morphisms to morphisms

- Preserves identity morphisms
- Preserves function composition

**Examples:**

- **Forgetful:** Group $\rightarrow$ Set sends groups to sets of elements
- **$C(c, - )$:** $C \rightarrow$ Set sends $x$ to set of morphisms $c \rightarrow x$ and morphisms $x \rightarrow y$ to $C(c,x) \rightarrow C(c,y)$ by postcomposition
- **Constant:** $C \rightarrow c$ sends every object in $C$ to $c$, every morphism to the identity on $c$
An indexing category $J$ of a certain shape

A functor $F$ assigning objects and morphism in $C$ to that shape

Diagram $F : J \to C$: 

- An indexing category $J$ of a certain shape
- A functor $F$ assigning objects and morphism in $C$ to that shape
Natural transformations $F \Rightarrow G$ of functors $F, G : C \to D$:

- A collection of morphisms called components $\alpha_c : Fc \to Gc$
- For all $f : c \to c'$, the diagram commutes

If the components are isomorphisms, we have a natural isomorphism $F \cong G$
Cones

A natural transformation between the constant functor $c : J \to C$ and the diagram $F : J \to C$.

The components $\lambda_j$ are called legs.
Universal Properties

A functor $F : \mathcal{C} \to \text{Set}$ is **representable** if there is an object $c$ in $\mathcal{C}$ so that $\mathcal{C}(c, -) \cong F$

- Recall $\mathcal{C}(c, -)$ takes an object $c'$ to the set of morphisms $c \to c'$

The functor $F$ encodes a **universal property** of $c$
A limit is a universal cone:

- There is a natural isomorphism $\text{C}(\; -, \text{lim} \ F) \cong \text{Cone}(\; -, \ F)$

- Morphisms $c \rightarrow \text{Lim} \ F$ are in bijection with cones with summit $c$ over $F$
Limits in Geometry

Product

Diagram shape

Product of spaces

Spaces
Limits in Geometry

Pullback

Diagram shape

Spaces

Fiber of $x = i(*)$
Category Theory is everywhere

- Mathematical objects and their functions belong to categories
- Maps between different types of objects/functions are functors
- Universal properties such as limits describe constructions like products and fibers
Reference

“Category Theory in Context” by Emily Riehl