Degree and Intersection Theory

AIRIKA YEE
University of Pennsylvania

Mentor: Artur B. Saturnino

The aim of this presentation is to prove the Jordan Brouwer Separation Theorem.
### Definitions

**Degree of a Map**

\[ \deg_2 f = \# f^{-1}(y) \mod 2 \]
DEFINITIONS

DEGREE OF A MAP

\[ \text{deg}_2 f = \# f^{-1}(y) \mod 2 \]

DIRECTIONAL MAP

Given a compact, connected manifold \( X \) and a smooth map \( f : X \rightarrow \mathbb{R}^n \), the directional map of any \( z \in \mathbb{R}^n \) not in the image of \( f(x) \) is defined as:

\[
u : X \rightarrow S^{n-1}\
\]

\[
u(x) = \frac{f(x) - z}{|f(x) - z|}\
\]
WINDING NUMBER

\[ W_2(f, z) = \text{deg}_2(u) \]
The complement of the compact, connected manifold $X$ consists of two connected open sets: the outside $D_0$ and an inside $D_1$. 
Any fixed point in $\mathbb{R}^n - X$ can be joined to a point in a neighborhood of some $x \in X$ without intersecting $X$. 

**STEP ONE** 
Any fixed point in $\mathbb{R}^n - X$ can be joined to a point in a neighborhood of some $x \in X$ without intersecting $X$. 

[JORDAN BROUWER SEPARATION THEOREM] 
\[ \mathbb{R}^n \setminus X = D_0 \cup D_1 \text{ with } D_0, D_1 \text{ disjoint.} \]
JORDAN BROUWER SEPARATION THEOREM

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**STEP TWO**

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$\mathbb{R}^n - X$ has, at most, 2 connected components. Points in the same connected component have the same winding number.

Homotopy between $z_0, z_1$ directional maps.

$$u_t(x) = \frac{x - z_t}{|x - z_t|}$$

Degree is invariant under homotopy.

$$\text{deg}_2(u_0) = \text{deg}_2(u_1)$$

$$W_2(X, z_0) = W_2(X, z_1)$$

**JORDAN BROUWER SEPARATION THEOREM**

$$\mathbb{R}^n \setminus X = D_0 \cup D_1$$ with $D_0, D_1$ disjoint.
Consider a ray, $r = \{z + t\bar{v} : t \geq 0\}$ that intersects $X$.

\[ W_2(X, z_0) = W_2(X, z_1) + l \mod 2 \]
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\[ u^{-1}(z_0, v) = \{ l_1, l_2, l_3, l_4 \} \]

\[ W_2(X, z_0) = W_2(X, z_1) + l \mod 2 \]
STEP FOUR

$R^n - X$ has precisely two components.

\[ D_0 = \{ z : W_2(X, z) = 0 \} \]

\[ D_1 = \{ z : W_2(X, z) = 1 \} \]
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JORDAN BROUWER SEPARATION THEOREM

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STEP FIVE

If \( z \) is very large, then \( W_2(X, z) = 0 \).
(i.e., \( D_0 \) is the “outside” of \( X \)).
We’ve shown that a simple, closed curve in $\mathbb{R}^n$ can be separated into an “inside” and “outside,” which can be identified by the mod 2 winding number.
The idea of a direction map is seen in the proof of other theorems.
Ex: Poincare-Hopf Theorem

• Really interesting results from counting points!
• Thank you Artur for the past two semesters in the Directed Reading Program!
THANK YOU