Statement of the Problem: A baseball pitcher throws a baseball of mass $m$. The force on the baseball is the sum of the force exerted by the pitcher together with a constant downward gravitational force equal to $(0, 0, -mg)$. The ball follows the path described by

$$r(t) = (t, t^2/2, 0) \quad \text{for} \quad 0 \leq t \leq 1$$

where $t$ is time. The pitcher can exert a force of magnitude at most $f$ in a direction perpendicular to the motion of the ball before he hurts his arm. Find a formula which determines from $m$, $g$ and $f$ whether this is a safe pitch for the pitcher. If the pitch is unsafe, will an injury occur at the beginning of the pitch (when $t = 0$) or near the end (when $t$ is close to 1)?
To solve this problem, let $F_p(t)$ be the total force vector the pitcher exerts on the ball. We can write

\[ F_p(t) = F_1(t) + F_2(t) \]  

with $F_1(t)$ the force in the direction of motion of the ball at time $t$ and $F_2(t)$ the component of $F(t)$ which is perpendicular to the direction of motion. The total force on the ball is then

\[ F(t) = F_p(t) + (0, 0, -mg) \]

since gravity also acts on the ball.

By Newton’s law,

\[ F(t) = m r''(t) \]

We can now use the decomposition of acceleration into tangential and normal components. From (0.4) and (0.5) the formula for this discussed in class becomes:

\[ r''(t) = s''(t)T(t) + \kappa(t)s'(t)^2 N(t) \]

\[ = s''(t)T(t) + (1 + t^2)^{-3/2} (1 + t^2) N(t) \]

From Newton’s Law (0.3), (0.6), (0.2) and (0.1) we get

\[ F(t) = m r''(t) \]

\[ = ms''(t)T(t) + m(1 + t^2)^{-1/2} N(t) \]

\[ = F_1(t) + F_2(t) + (0,0,-mg) \]

Here $F_1(t)$ is the force the pitcher exerts in the direction of motion, and this is a multiple of the unit tangent vector $T(t) = (1,t,0)/\sqrt{1+t^2}$. The vector $N(t)$ is perpendicular to $T(t)$. The vector $F_2(t)$ is perpendicular to $T(t)$ since this is the force the pitcher exerts perpendicular to the direction of motion. The vector $(0,0,-mg)$ is also perpendicular to $T(t)$, as one can see by taking a dot product or by simply from the fact that gravity acts in the negative $z$ direction, which is perpendicular to any vector in the $x-y$ plane.

We now equate the components of $F(t)$ in the last two equations of (0.7) which are perpendicular to $T(t)$. This gives

\[ m(1 + t^2)^{-1/2} N(t) = F_2(t) + (0,0,-mg) \]

Thus $F_2(t)$ is

\[ F_2(t) = m(1 + t^2)^{-1} N(t) + (0,0,mg) \]

We are interested in the magnitude of $F_2(t)$. The vector $T(t) = r'(t) / |r'(t)|$ has $z$-component equal to 0. On taking derivatives of the components of $T(t)$, we see that $T'(t)$ also has $z$-component equal to 0. So $N(t) = T'(t) / |T'(t)|$ also has $z$-component equal to 0. Therefore $N(t)$ lies in the
$x$-$y$-plane, so $N(t)$ is perpendicular to $(0, 0, mg)$. This means that we can apply the pythagorean theorem to find the length of $F_2(t)$ in (0.9). This gives

$$|F_2(t)|^2 = |m(1 + t^2)^{-1}N(t)|^2 + |(0, 0, mg)|^2$$

(0.10)

$$= m^2(1 + t^2)^{-2}|N(t)|^2 + (mg)^2$$

$$= m^2((1 + t^2)^{-2} + g^2)$$

where we have $|N(t)| = 1$ since $N(t)$ is a unit vector.

Taking square roots in (0.10) shows

$$|F_2(t)| = m\sqrt{(1 + t^2)^{-2} + g^2}$$

(0.11)

This will be largest for $t = 0$, so

$$\max\{|F_2(t)| : 0 \leq t \leq 1\} = |F_2(0)| = m\sqrt{1 + g^2}$$

(0.12)

So the pitch will be safe provided

$$m\sqrt{1 + g^2} \leq f = \text{maximum safe force perpendicular to motion}$$

(0.13)

If the pitch is not safe, (0.12) shows that $|F_2(0)|$ will exceed $f$. So the most dangerous moment for the pitcher is in fact $t = 0$, right when the pitch starts.