MATH 114 - EXTRA CREDIT PROBLEM # 3

Abstract. You can turn in this problem anytime before the final exam. Work on extra credit problems will be used to decide cases in which a course grade is in between two letter grades.

Precession of spinning wheels II

This problem continues extra credit problem #2, which you should do first. Rather than just considering how the wheel begins to move when one end of the axle is released at time $t = 0$, we want to analyze the angular momentum of a wheel which is precessing at a constant speed. Then we'll check that this agrees with a computation of gravitational torque similar to the one in problem # 2.

We will now suppose that one end of the axle of the wheel is at the origin, and the center of the axle is at

$$E(t) = a(\cos(\nu t), \sin(\nu t), 0)$$

(0.1)

at time $t$, where $\nu$ is the angular velocity of the axle about the origin in the $x - y$ plane. As in problem #2, we will suppose that the wheel spins vertically about the axle at a constant angular velocity of $\omega$ radians per second, in a counterclockwise direction when one looks down the axle toward the origin and through the spokes of the wheel. The radius of the wheel is $R$ and the distance from the origin to the center of the axle is $a$.

\[\begin{align*}
\text{a. Show that the vector} \\
\mathbf{r}(t, \theta) &= a(\cos(\nu t), \sin(\nu t), 0) + R \cos(\omega t + \theta) \cdot (-\sin(\nu t), \cos(\nu t), 0) + R \sin(\omega t + \theta) \cdot (0, 0, 1) \\
\text{describes the position of the part of the rim at time } t \text{ which at time } t = 0 \text{ is at position } r(0, \theta).
\end{align*}\]

Hint: The first term on the right side of (0.2) gives the position of the center of the axle at time $t$. Given this position, consider how to get from it to the point on the rim we are interested in. This displacement is a sum of a vector in the $x - y$ plane which is perpendicular
to the vector from the origin to the center of the axle plus a vertical component in the z-direction. Write down the unit vectors in these directions, and then multiply them by the lengths which are appropriate to how far the wheel has rotated around the center of the axle at time $t$.

b. Find the velocity vector

$$ v(t, \theta) = \frac{d}{dt} r(t, \theta) $$

at time $t$ of the part of the rim which is at $r(t, \theta)$ by differentiating the components of $r(t, \theta)$ with respect to $t$ (and keeping $\theta$ fixed).

c. Suppose $\Delta \theta$ is some small positive number. If the wheel has a constant density $\rho$, the mass of the small piece of the wheel between positions $r(t, \theta)$ and $r(t, \theta + d\theta)$ is approximately

$$ m = \rho R \Delta \theta $$

since the length of this piece is approximately $R \Delta \theta$.

For small $\Delta \theta$, the angular momentum of this piece is approximately

$$ \vec{A}(t, \theta, \Delta \theta) = m \left( r(t, \theta) \times v(t, \theta) \right) = (r(t, \theta) \times v(t, \theta)) \rho R \Delta \theta $$

Subdivide the wheel into small pieces each of which have length $R \Delta \theta$. Add up the angular momentum vectors for each of these small pieces, and let $\Delta \theta \to 0$. This leads to an integral which gives the total angular momentum vector of the wheel at time $t$:

$$ \vec{\Delta}(t) = \int_{\theta_0}^{\theta_0 + 2\pi} (r(t, \theta) \times v(t, \theta)) \rho R d\theta $$

Here part (a) and part (b) of the problem express $r(t, \theta)$ and $v(t, \theta)$ as linear combinations of certain vectors. We can use these combinations to write out $r(t, \theta) \times v(t, \theta)$ as a linear combination of cross products of certain vectors. To evaluate (0.4), take advantage of the fact that for all $t$

$$ \int_{\theta_0}^{2\pi} \cos(\omega t + \theta) d\theta = \int_{\theta_0}^{2\pi} \sin(\omega t + \theta) d\theta = \int_{\theta_0}^{2\pi} \cos(\omega t + \theta) \sin(\omega t + \theta) d\theta = 0 $$

while

$$ \int_{\theta_0}^{2\pi} \cos^2(\omega t + \theta) d\theta = \int_{\theta_0}^{2\pi} \sin^2(\omega t + \theta) d\theta = \pi $$

Using this you can greatly simplify the calculation of $\vec{\Delta}(t)$. Does $\vec{\Delta}(t)$ point along the axis of the wheel? Resolve $\vec{\Delta}(t)$ into the sum of a vector pointing along the axis of the wheel with a vector perpendicular to the axis.

Comment: In computing the angular momentum of a precessing wheel, many physics texts ignore the fact that the axis of wheel is in fact moving.

d. Carry out the computation of total gravitational torque $\tau(t)$ on the wheel at time $t$ using the constant of gravity $g$ and calculations similar to those in extra credit problem # 2. Then set

$$ \tau(t) = \frac{d}{dt} \left( \vec{\Delta}(t) \right) $$

What relationship should exist between the constants $g, \rho, R, \omega$ and $\nu$? Show that the angular velocity of precession $\nu$ is determined by the other constants, as expected. What happens to $\nu$ as $\omega$ is increased? Does this make physical sense?