MATH 203, PROBLEM SET 2

DUE IN LECTURE ON FRIDAY, JAN. 27.

1. Suppose $A = \{a_1, a_2\}$ is a set with two elements. How many relations between $A$ and $A$ are there? How many of these are partial orders? How many are total orders?

**Extra Credit:** Answer the same questions when $A = \{a_1, a_2, a_3\}$ has 3 elements. I recommend trying this only after you’ve finished all the other problems.

2. Suppose $A = \{a_1, a_2, \ldots , a_n\}$ is a finite set with $n > 0$ distinct elements. Let $\leq$ be a total order on $A$.

   i. Use induction on $n$ to show that there is a smallest element $s(A)$ of $A$. Recall that this means $s(A) \leq x$ for all $x \in A$.

   **Hints:** If $A' = \{a_1, \ldots, a_n, a_{n+1}\}$ and $A = \{a_1, \ldots, a_n\}$, what should $s(A')$ be in terms of $a_{n+1}$ and $s(A)$?

   ii. Use induction on $n$ to show that there are exactly $n! = n \cdot (n-1) \cdots 2 \cdot 1$ possible total orders $\leq$ on $A$.

   **(Hint:** For the induction step, suppose $A' = \{a_1, \ldots, a_n\}$ has $n+1$ elements. Show that there is a bijection between the set of total orders $\leq'$ on $A'$ and the set of pairs $(x, \leq)$ in which $x$ is an element of $A'$ and $\leq$ is a total order on $A' - \{x\}$, where $A' - \{x\}$ has $n$ elements. This bijection can be arranged so that $x$ is the smallest element of $A'$ with respect to $\leq'$.)

3. Show that if $D$ is an infinite subset of a model $\mathbb{N}$ of the integers then $D$ is countable.

   **Hints:** Use a bijection between $\mathbb{N}$ and the standard model $\mathbb{Z}^+ = \{1, 2, 3, \ldots \}$ of the positive integers to show it is enough to consider the case in which $\mathbb{N}$ is $\mathbb{Z}^+$. You need to show there is a bijection $f : \mathbb{Z}^+ \rightarrow D$ given that $D$ is an infinite subset of $\mathbb{Z}^+$. Use induction to show that for each $n$ in $\mathbb{Z}^+$, there is a map $f_n : \{1, \ldots, n\} \rightarrow D$ such that $f_1(1)$ is the smallest element of $D$, $f_{n-1}(m) = f_n(m)$ if $n > 1$ and $m \in \{1, \ldots, n-1\}$, and $f_n(n)$ is the smallest element of $D - \{f_{n-1}(1), \ldots, f_{n-1}(n-1)\}$ if $n > 1$. Then use these $f_n$’s to define $f$. You can use the fact that $\mathbb{Z}^+$ is well ordered, which was proved in class.

4. Show that the product set $(\mathbb{Z}^+)^2 = \{(a, b) : a, b \in \mathbb{Z}^+\}$ has the same cardinality as $\mathbb{Z}^+$.

   **Hints:** Show that for $n = 1, 2, 3, \ldots$, there is a bijection $f_n$ from $\{1, \ldots, n(n+1)\}$ to the set $T_n = \{(a, b) \in \mathbb{Z}^+, a + b \leq n + 1\}$ such that $f_1(1) = (1, 1)$ and $f_n$ agrees with $f_{n-1}$ on $\{1, \ldots, \frac{n(n-1)}{2}\}$ if $n > 1$. To see what is going on, plot the points in $T_n$ for some small values of $n$. You can use without proof the identity $\sum_{j=1}^{n} = \frac{n(n+1)}{2}$ which we proved in class. Then use these $f_n$ to define a bijection $f : \mathbb{Z}^+ \rightarrow (\mathbb{Z}^+)^2$.

5. Show the set $\mathbb{Q}^+$ of rational numbers is countable.
**Extra Credit:** Show that the set $\mathbb{Q}$ of all rationals is countable.

**Hints:** You can use without proof that every positive rational has the form $\frac{a}{b}$ for a unique ordered pair $(a, b)$ of positive integers. Show this identifies $\mathbb{Q}^+$ with an infinite subset of $(\mathbb{Z}^+)^2$, and the use problems 3 and 4 above.